ISRAEL JOURNAL OF MATHEMATICS 107 (1998), 157–193

CONVEX UNCONDITIONALITY AND SUMMABILITY OF WEAKLY NULL SEQUENCES

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ABSTRACT

It is proved that every normalized weakly null sequence has a subsequence which is convexly unconditional. Further, a hierarchy of summability methods is introduced and with this we give a complete classification of the complexity of weakly null sequences.

Introduction

In the present paper we investigate the behavior of the subsequences of a weakly null sequence $(x_n)_{n\in\mathbf{N}}$ of a Banach space X with respect to two fundamental properties. The first is convex unconditionality which is investigated in the first section of the paper. This is defined as:

Definition: A normalized sequence $(x_n)_{n\in\mathbf{N}}$ in a Banach space X is said to be **convexly unconditional** if for every $\delta > 0$ there exists $C(\delta) > 0$ such that if an absolutely convex combination $x = \sum_{n=1}^{\infty} a_n x_n$ satisfies $\|x\| > \delta$ then $\|\sum_{n=1}^{\infty} \varepsilon_n a_n x_n\| > C(\delta)$ for every choice of signs $(\varepsilon_n)_{n\in\mathbf{N}}$.

The result we prove here is the following theorem.

THEOREM A: *If $(x_n)_{n\in\mathbf{N}}$ is a normalized weakly null sequence in a Banach space X then it has a convexly unconditional subsequence.*

A fundamental example due to B. Maurey and H. Rosenthal [M–R] showed that we could not expect that every normalized weakly null sequence has an

Received October 7, 1996

unconditional subsequence. The recent examples [G–M], [A–D] show that there are spaces without any unconditional basic sequence. On the other hand there are results where some weaker forms of unconditionality appear. One of them is due to J. Elton [E], [O_1] which is related to the unconditional behavior of linear combinations with coefficients bounded away from zero, and the other is due to E. Odell [O_2] and it is related to the unconditionality of Schreier admissible linear combinations. Our theorem is in the same direction as Elton's Theorem; more precisely it is the dual result, and the proof is based, as is his proof, on infinite Ramsey Theory. The result follows from a combinatorial principle (Lemma 1.2) which seems to be of independent interest and which is also used in the second part of the paper.

The existence of a convexly unconditional sequence is strong evidence that the convex bounded sets behave much better to unconditionality than the subspaces of a Banach space.

In the second part we deal with summability methods. The starting point for our investigation is the following question.

It follows from Mazur's theorem that every weakly null sequence has convex combinations norm converging to zero. The general question is to describe "regular" convex combinations with this property. This problem dates to the early days of the development of Banach space theory. Banach and Saks proved that every bounded sequence in $L^p(\mu)$, $1 < p < \infty$ has a norm Cesaro summable subsequence. This result was extended by W. Szlenk for weakly convergent sequences in $L^1(\mu)$. Shortly after the Banach–Saks Theorem, an example was given by J. Schreier [Sch] of a weakly null sequence with no norm Cesaro summable subsequence. Schreier's example is defined as follows: First we define the following family,

$$\mathcal{F} = \{F \subset \mathbf{N}\colon \#F \leq \min F\} \cup \{\emptyset\}.$$

Then on the vector space $c_{00}(\mathbf{N})$ of eventually zero sequences of reals we define the norm

$$\|(a_n)_{n\in\mathbf{N}}\| = \sup\left\{\sum_{n\in F} |a_n|\colon F \in \mathcal{F}\right\}.$$

It is easy to see that $\mathcal{F}$ is compact in the topology of pointwise convergence. Hence the unit vector basis $(e_n)_{n\in\mathbf{N}}$ is weakly null. Further, from the definition of $\mathcal{F}$ we get that for every k and integers $n_1 < n_2 < \cdots < n_k$ we have

$$\left\|\frac{e_{n_1} + e_{n_2} + \cdots + e_{n_k}}{k}\right\| \geq \frac{1}{2}.$$

So no subsequence of $(e_n)_{n\in\mathbf{N}}$ is norm Cesaro summable. Later it was proved by H. Rosenthal that if $(x_n)_{n\in\mathbf{N}}$ is weakly null and no subsequence is norm Cesaro summable then there exists $(n_i)_{i\in\mathbf{N}}$ and $\epsilon > 0$ such that

$$\|\sum_{i\in F} a_i e_{n_i}\| > \epsilon \cdot \sum_{i\in F} |a_i|$$

for all $F \in \mathcal{F}$. Whenever this property appears, we say that the sequence $(x_{n_i})_{i\in\mathbf{N}}$ is an ℓ^1 spreading model. This result, in connection with a theorem proved by P. Erdös and M. Magidor [E–M], gives the following dichotomy.

THEOREM: *For every weakly null sequence $(x_n)_{n\in\mathbf{N}}$ exactly one of the following holds:*

(a) *For every $M \in [\mathbf{N}]$ there exists $L \in [M]$ such that for all $P \in [L]$, $P = (n_i)_{i\in\mathbf{N}}$, the subsequence $(x_{n_i})_{i\in\mathbf{N}}$ is norm Cesaro summable.*
(b) *There exists $M \in [\mathbf{N}]$, $M = (m_i)_{i\in\mathbf{N}}$ such that the subsequence $(x_{m_i})_{i\in\mathbf{N}}$ is an ℓ^1 spreading model.*

A proof of this theorem is also given in [M].

This theorem is complete when condition (a) holds. If (b) holds then there is no information on the structure of convex combinations that converge in norm to zero. Our aim is to give a full extension of the above theorem and through this to describe the complexity of weakly null sequences. For this we use two hierarchies, the **Schreier Hierarchy** and the **Repeated Averages Hierarchy.**

THE SCHREIER HIERARCHY. The Schreier family $\mathcal{F}$ is quite important in the theory of Banach spaces. Recall that it is one of the main ingredients in the definition of Tsirelson's space [T]. D. Alspach and S. Argyros [Al–Ar] defined a family $\{\mathcal{F}_\xi\}_{\xi<\omega_1}$ called generalized Schreier families. The definition of $\mathcal{F}_\xi$ is given in the following way:

Set $\mathcal{F}_0 = \{\{n\}: n \in \mathbf{N}\} \cup \{\emptyset\}$ and $\mathcal{F}_1 = \mathcal{F}$.

If $\mathcal{F}_\xi$ has been defined then we set

$$\mathcal{F}_{\xi+1} = \left\{\bigcup_{i=1}^{n} F_i: n \leq F_1 < F_2 < \cdots < F_n, F_i \in \mathcal{F}_\xi\right\} \cup \{\emptyset\}.$$

If ξ is a limit ordinal choose $(\xi_n)_{n\in\mathbf{N}}$ strictly increasing to ξ and set

$$\mathcal{F}_\xi = \{F: F \in \mathcal{F}_{\xi_n}, n \leq \min F\} \cup \{\emptyset\}.$$

We call this family the Schreier Hierarchy since it carries certain strong universal properties some of which are described in the present paper. Roughly speaking, the complexity of every compact countable metric space is dominated by some member of $\{\mathcal{F}_\xi\}_{\xi<\omega_1}$. Further members of $\{\mathcal{F}_\xi\}_{\xi<\omega_1}$ appear naturally in several cases. For example, the n^{th} norm in the inductive definition of Tsirelson's space is implicitly connected to the family $\mathcal{F}_n$. Explicitly the family $(\mathcal{F}_n)_{n\in\mathbf{N}}$ appeared for the first time in an example constructed by E. Odell [A–O]. Recently $\{\mathcal{F}_\xi\}_{\xi<\omega_1}$ have been used in the investigation of asymptotic ℓ^p spaces. Connected to the family $\{\mathcal{F}_\xi\}_{\xi<\omega_1}$ is the following definition.

Definition: Let $(x_n)_{n\in\mathbf{N}}$ be a bounded sequence in a Banach space X. For $M \in [\mathbf{N}]$, $M = (m_i)_{i\in\mathbf{N}}$ we say that $(x_{m_i})_{i\in\mathbf{N}}$ is an ℓ^1_ξ spreading model if there exists $\epsilon > 0$ such that for all choices, $(a_i)_{i\in F}$, for $F \in \mathcal{F}_\xi$ we have that:

$$\|\sum_{i\in F} a_i x_{m_i}\| \geq \epsilon \cdot \sum_{i\in F} |a_i|.$$

It is clear that an ℓ^1_1 spreading model is the usual ℓ^1 spreading model. Since the families $(\mathcal{F}_\xi)_{\xi<\omega_1}$ are of increasing complexity, the existence of a subsequence which is an ℓ^1_ξ model, for large ξ, describes strong ℓ^1 behavior of the given sequences. As proved in [Al–Ar], if a sequence contains ℓ^1_ξ spreading models for all $\xi < \omega_1$ then actually it contains a subsequence equivalent to the unit vector basis of ℓ^1.

The second hierarchy introduced here is that of Repeated Averages.

THE REPEATED AVERAGES HIERARCHY. To introduce this we give some notations and definitions.

We denote by $S^+_{\ell^1}$ the positive part of the unit sphere of $\ell^1(\mathbf{N})$. If $H = (x_n)_{n\in\mathbf{N}}$ is a bounded sequence in a Banach space and $A = (a_n)_{n\in\mathbf{N}} \in S^+_{\ell^1}$ we set

$$A \cdot H = \sum_{n=1}^{\infty} a_n x_n \in X.$$

For $M \in [\mathbf{N}]$ a sequence $(A_n)_{n\in\mathbf{N}}$ of successive blocks in $S^+_{\ell^1}$ defines an M-summability method, denoted by $M{-}(A_n)_{n\in\mathbf{N}}$ if $M = \bigcup_{n=1}^{\infty} \operatorname{supp} A_n$. It is clear that $M - (A_n)_{n\in\mathbf{N}}$ is a regular summability method in the classical sense (cf. [K] pp. 480–481).

Definition: A sequence $H = (x_n)_{n\in\mathbf{N}}$ in a Banach space is $M{-}(A_n)_{n\in\mathbf{N}}$ summable if the sequence $(A_n \cdot H)_{n\in\mathbf{N}}$ is Cesaro summable.

The RA Hierarchy is defined, inductively, for every $M \in [\mathbf{N}]$ and $\xi < \omega_1$ and it is an M-summability method denoted by $(\boldsymbol{\xi}^M_n)_{n\in\mathbf{N}}$. We also use the notation (M, ξ) for the same method. Thus the RA Hierarchy is the family

$$\{(M, \xi)\colon M \in [\mathbf{N}], \xi < \omega_1\}.$$

The precise definition is given at the beginning of the second section of the paper. A brief desciption of it goes as follows: For $\xi = 0$ and $M = (m_n)_{n\in\mathbf{N}}$ we set $\boldsymbol{\xi}_n^M = e_{m_n}$. Thus the (M,ξ)-summability, for $\xi = 0$, of a weakly null sequence $(x_n)_{n\in\mathbf{N}}$ is exactly the norm Cesaro summability of the subsequence $(x_{m_n})_{n\in\mathbf{N}}$ where $M = (m_n)_{n\in\mathbf{N}}$.

If $(\boldsymbol{\xi}_n^M)_{n\in\mathbf{N}}$ has been defined then for $\zeta = \xi + 1$ we set $\boldsymbol{\zeta}_n^M$ to be the average of an appropriate number of successive elements of $(\boldsymbol{\xi}_n^M)_{n\in\mathbf{N}}$. This justifies the term Repeated Averages. For ζ a limit ordinal $(\boldsymbol{\zeta}_n^M)_{n\in\mathbf{N}}$ is constructed by a careful choice of terms of $\left\{(\boldsymbol{\xi}_n^M) \colon \xi < \zeta, n \in \mathbf{N}\right\}$.

One property we would like to mention here is that $\operatorname{supp}\boldsymbol{\xi}_n^M \in \boldsymbol{\mathcal{F}}_\xi$ and moreover it is, in a sense, a maximal element of $\boldsymbol{\mathcal{F}}_\xi$. Thus $(\boldsymbol{\xi}_n^M)_{n\in\mathbf{N}}$ exhausts the complexity of the family $\boldsymbol{\mathcal{F}}_\xi$. More important is that $(\boldsymbol{\xi}_n^M)_{n\in\mathbf{N}}$ carries some nice stability properties (see P.3 – P.4 after the precise Definition in Section 2) which allows us to handle them in the proofs of the theorems.

The difference between the RA Hierarchy and the summability methods described as an infinite matrix is that in the RA Hierarchy the summability of a subsequence $(x_n)_{n\in M}$ depends on the subset M while in the usual case, after reordering $(x_n)_{n\in M}$ as $(x_{n_k})_{k\in\mathbf{N}}$, we ignore the set M and apply the summability method with respect to the index k. Thus in our case for a fixed countable ordinal ξ we have 2^ω methods $\{(M,\xi) \colon M \in [\mathbf{N}]\}$ which have uniformly bounded complexity. This is so, since for every $M \in [\mathbf{N}]$, $n \in \mathbf{N}$, the set $\operatorname{supp}\boldsymbol{\xi}_n^M$ belongs to the compact family $\boldsymbol{\mathcal{F}}_\xi$.

For a given $M \in [\mathbf{N}]$ the methods $\{(M,\xi) \colon \xi < \omega_1\}$ are increasing very fast. It is worthwhile to remark that if for $\xi < \omega_1$ and $n \in \mathbf{N}$ we set $k_n^\xi = \min\operatorname{supp}\xi_n^{\mathbf{N}}$ then the family $\{(k_n^\xi) \colon n \in \mathbf{N}, \xi < \omega_1\}$ is the Ackermann Hierarchy, a well known hierarchy of Mathematical Logic.

THEOREM B: *For a weakly null sequence $(x_n)_{n\in\mathbf{N}}$ in a Banach space X and $\xi < \omega_1$ exactly one of the following holds.*

(a) *For every $M \in [\mathbf{N}]$ there exists $L \in [M]$ such that for every $P \in [L]$ the sequence $(x_n)_{n\in\mathbf{N}}$ is (P,ξ) summable.*

(b) *There exists $M \in [\mathbf{N}]$ $M = (m_i)_{i\in\mathbf{N}}$ such that $(x_{m_i})_{i\in\mathbf{N}}$ is an $\ell^1_{\xi+1}$ spreading model.*

It is proved in [Al–Ar] that for every weakly null sequence $(x_n)_{n\in\mathbf{N}}$ there exists $\xi < \omega_1$ such that for every $\zeta \geq \xi$ no subsequence of $(x_n)_{n\in\mathbf{N}}$ is an ℓ^1_ζ spreading model. So we introduce the **Banach–Saks** index of a weakly null sequence defined as

$$\mathrm{BS}[(x_n)_{n\in\mathbf{N}}] = \min\{\xi\colon \text{ no subsequence of } (x_n)_{n\in\mathbf{N}} \text{ is an } \ell^1_\xi \text{ spreading model}\}$$

and from Theorem B we get the following

THEOREM C: *Let $H = (x_n)_{n\in\mathbf{N}}$ be a weakly null sequence with $\mathrm{BS}[(x_n)_{n\in\mathbf{N}}] = \xi$. Then ξ is the unique ordinal satisfying the following:*

(a) *For every $M \in [\mathbf{N}]$ there exists $L \in [M]$ such that for every $P \in [L]$, $\lim_{n\in\mathbf{N}} \|\boldsymbol{\xi}^P_n \cdot H\| = 0$.*
(b) *For every $\zeta < \xi$ there exists $L_\zeta \in [\mathbf{N}]$ such that $L_\zeta = (n_i)_{i\in\mathbf{N}}$ and $(x_{n_i})_{i\in\mathbf{N}}$ is an ℓ^1_ζ spreading model.*
(c) *If $\xi = \zeta + 1$ there exists $\epsilon > 0$ and $L \in [\mathbf{N}]$ such that for all $P \in [L]$, $\left\|\boldsymbol{\zeta}^P_n \cdot H\right\| > \epsilon$ and $\left(\boldsymbol{\zeta}^P_n \cdot H\right)_{n\in\mathbf{N}}$ is Cesaro summable.*

For $\xi = 0$ Theorem B implies exactly the dichotomy mentioned at the beginning of the introduction (Theorem). Theorem C gives the full description of the norm summability for a weakly null sequence in terms of the methods $\{(M,\xi)\colon M \in [\mathbf{N}], \xi < \omega_1\}$. This justifies the universal character of these summability methods as well as the universal character of the Schreier Hierarchy since, as we mentioned above, the $\operatorname{supp}\boldsymbol{\xi}^M_n$ belongs to $\mathcal{F}_\xi$.

Definition: (a) A Banach space X has the **ξ-Banach–Saks property** (ξ-BS) if for every bounded sequence $(x_n)_{n\in\mathbf{N}}$ in X there exists $L \in [\mathbf{N}]$ such that $(x_n)_{n\in\mathbf{N}}$ is (L,ξ) summable.

(b) The space X has the **weak ξ-Banach–Saks property** (w ξ-BS) if the above property holds for weakly convergent sequences.

The next corollaries follow from Theorems B and C.

COROLLARY: *For every separable reflexive Banach space X there exists a unique ordinal $\xi < \omega_1$ such that:*

(i) *For all ordinals $\zeta \geq \xi$ the space X has ζ-BS.*
(ii) *For every $\zeta < \xi$ the space X fails ζ-BS.*

COROLLARY: *If X is a separable Banach space not containing ℓ^1 isomorphically then there exists a unique ordinal $\xi < \omega_1$ such that:*

(i) *For all ordinals $\zeta \geq \xi$ the space X has w ζ-BS.*
(ii) *For every $\zeta < \xi$ the space X fails w ζ-BS.*

In the final part we introduce an ordinal index, the anti-uniform convergence index (auc-index) which is connected to the existence of ℓ^1_ξ-spreading models. We prove the following.

THEOREM D: *If $(x_n)_{n\in \mathbf{N}}$ is a weakly null sequence and its auc index is greater than ω^ξ then there exists $M \in [\mathbf{N}]$ such that $(x_n)_{n\in M}$ is an ℓ^1_ξ spreading model.*

The proofs of the above theorems use infinite Ramsey theory and an index introduced here for compact families of infinite subsets of $\mathbf{N}$ that is called the strong Cantor Bendixson index. This index helps us to develop a criterion for embedding the family $\mathcal{F}_\xi$ into a family $\mathcal{F}$ provided the index of $\mathcal{F}$ is greater than ω^ξ. Also, the proofs of these theorems make use of Lemma 1.2 and a variation of it.

ACKNOWLEDGEMENT: The authors express their thanks to G. Androulakis and E. Odell for their useful suggestions.

NOTATION. For an infinite subset N of $\mathbf{N}$ we denote by $[N]$ the set of all infinite subsets of N. Further, we denote by $[N]^{<\omega}$ the set of all finite subsets of the set N. In the sequel for $F \in [\mathbf{N}]^{<\omega}$ we will identify the set F with its characteristic function. Thus for $A = (a_n)_{n\in\mathbf{N}}$ in $\ell^1(\mathbf{N})$ we will denote by $\langle A, F\rangle$ the quantity $\sum_{n\in F} a_n$. For $M \in [\mathbf{N}]$ we will denote by $M = (m_i)_{i\in\mathbf{N}}$ the natural order of the set M. We topologize $[\mathbf{N}]$ by the topology of the pointwise convergence, by considering $[\mathbf{N}]$ as a subset of the space $\mathbf{N}^{\mathbf{N}}$ of irrationals.

As we mentioned above, our proofs use in an essential way an infinite Ramsey Theorem. This theorem, one of the most important principles in infinite combinatorics was proved in several steps by Nash-Williams [N-W], Galvin and Prikry [G–P] and in the final form by Silver [Si]. Silver's proof was model-theoretic. Later Ellentuck [Ell] gave a proof of Silver's result using classical methods. We recall the statement of the theorem.

0.1. THEOREM: *Let A be an analytic subset of $[\mathbf{N}]$. Then for every $M \in [\mathbf{N}]$ there exists $L \in [M]$ such that either $[L] \subset A$ or else $[L] \subset [M] \setminus A$.*

In the sequel any set A satisfying the above property will be called completely Ramsey.

1. Convex unconditionality

The first section is devoted to the proof of Theorem 1.3 and some consequences of it. Our approach is similar to Elton's proof of the result that we mentioned in the introduction, in particular, Lemmas 1.1, 1.2 and the dual statements of the corresponding results in his proof. As a consequence of Lemma 1.2 we get Corollary 1.6 about adequate subfamilies of campact families of finite sets.

1.1. LEMMA: *Let F be a relatively weakly compact subset of $c_0(\mathbf{N})$. Then for every $N' \in [\mathbf{N}]$ there exists $M \in [N']$ such that: If $l_1 < l_2 < \cdots < l_n$ are elements of M and there exists $f \in F$ such that for every $i = 2, \ldots, n$, $f(l_i) > \delta$, then there exists $g \in F$ such that for every $i = 2, \ldots, n$, $g(l_i) > \delta$ and $|g(l_1)| < \epsilon$.*

Proof: For $n \in \mathbf{N}$ we set

$$\begin{aligned} S_n = \{M \in [N']\colon & M = (m_i) \text{ and if there exists } f \in F \text{ such that} \\ & \forall i = 2, \ldots, n, f(m_i) > \delta \text{ then there exists } g \in F \text{ such that} \\ & \forall i = 2, \ldots, n, g(m_i) > \delta \text{ and } ag(m_1)| < \epsilon\}. \end{aligned}$$

It is clear that each S_n is closed in the topology of pointwise convergence. Hence $S = \bigcap_{n=1}^{\infty} S_n$ is closed and therefore completely Ramsey. Choose $M \in [N']$ such that either $[M] \subset S$ or $[M] \subset [N'] \setminus S$.

Suppose that $[M] \subset [N'] \setminus S$. Let $M = (m_i)$ and consider any $n \in \mathbf{N}$. For $1 \leq j \leq n$ set

$$L_j = \{m_j, m_{n+1}, m_{n+2}, \ldots\},$$

which does not belong to S. Therefore, for any such j there are $f_j \in F$ and $l_j \in \mathbf{N}$ such that for all $i = 1, \ldots, l_j$, $f_j(m_{n+i}) > \delta$ and every $g \in F$ with $g(m_{n+i}) > \delta$ for $i \leq l_j$ satisfies $|g(m_j)| \geq \epsilon$. Set $l_{j_0} = \max\{l_j\colon 1 \leq j \leq n\}$ and $f^n = f_{j_0}$. Observe that for $1 \leq j \leq n$, $f^n(m_{n+i}) > \delta$ for all $i = 1, \ldots, l_j$ and hence $f^n(m_j) \geq \epsilon$ for all $j = 1, \ldots, n$. It is clear now that the sequence $\{f^n\}$ does not have a weakly convergent subsequence. Therefore the case $[M] \subset [N'] \setminus S$ is impossible and it is easy to check that if $[M] \subset S$ then M satisfies the conclusion of the lemma.

1.2. LEMMA: *Consider a relatively weakly compact subset F of $c_0[\mathbf{N}]$, $\delta > 0$ and $0 < \epsilon < 1$. Then for every $N' \in [\mathbf{N}]$ there exists $M = (m_i) \in [N']$ such that:*

For every $f \in F$, $n \in \mathbf{N}$ and $I \subset \{1, \ldots, n\}$ with $\min_{i \in I} f(m_i) \geq \delta$ there exists $g \in F$ satisfying the following two conditions:

(i) $\min_{i \in I} g(m_i) > (1 - \epsilon)\delta$,

(ii) $\sum_{\{i \leq n\colon i \notin I\}} |g(m_i)| < \epsilon \cdot \delta$.

Proof: Choose $a > 0$ with $|f(m)| \leq a$ for $m \in \mathbf{N}$, $f \in F$. Next we choose a strictly increasing sequence (k_n) of natural numbers such that $2^{k_1} > a$ and if $\epsilon_n = 1/2^{k_n}$ then $\sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \epsilon_k < \epsilon \cdot \delta$.

We divide the proof into two stages. In the first we will construct the set M and in the second we will show that it satisfies the conclusion of the lemma.

The set $M = (m_i)$ is defined inductively so that the following conditions are fulfilled.

If I is a finite subset of $\mathbf{N}$, and $j < \min I$, then for every $f \in F$ such that $\min_I f(m_i) > \delta$ there exists $g \in F$ with:

(a) $\min_I g(m_i) > \delta$,

(b) $|g(m_j)| < \epsilon_j$,

(c) $|g(m_i) - f(m_i)| \leq \epsilon_j$ for $i = 1, 2, \ldots, j-1$.

To find such an M we choose inductively a decreasing sequence of infinite sets $N' \supset N_1 \supset \cdots \supset N_i \supset \cdots$ and we set $m_i = \min N_i$.

To choose N_1 we apply Lemma 1.1 to find a subset N_1 of N' such that the conclusion of Lemma 1 holds for the given δ and $\epsilon = \epsilon_1$. This finishes the choice of N_1.

Suppose that $N' \supset N_1 \supset \cdots \supset N_j$ have been chosen such that if $m_i = \min N_i$ then $m_1 < m_2 < \cdots < m_j$ and, if $1 < i \leq j$, I is a finite subset of N_i with $m_i < \min I$, $f \in F$ with $\min_{k \in I} f(k) > \delta$ then there exists $g \in F$ satisfying (a), (b), (c). To choose N_{j+1} we consider the set W of all closed dyadic intervals of length $\epsilon_{j+1} = 1/2^{k_{j+1}}$ which are contained in the interval $[-2^{k_1}, 2^{k_1}]$. We denote by W^j the j-times product of W and for every $\mathcal{B} \in W^j$, $\mathcal{B} = (B_1, \ldots, B_j)$, we set

$$F_{\mathcal{B}} = \{f \in F: f(m_i) \in B_i, i \leq j\}$$

which clearly is relatively weakly compact.

Applying repeatedly Lemma 1.1, we find an infinite subset N_{j+1} of N_j such that $m_j < \min N_{j+1}$ and the conclusion of Lemma 1.1 holds for the set N_{j+1} and for every $F_{\mathcal{B}}$, $\mathcal{B} \in W^j$, the given δ and $\epsilon = \epsilon_{j+1}$. This completes the inductive construction of the sets (N_i) and hence the set M is defined.

It remains to show that M satisfies the desired properties.

The set M satisfies (i) and (ii). Given $n \in \mathbf{N}$, a subset I of $\{1, 2, \ldots, n\}$ and $f \in F$ such that $\min_{i \in I} f(m_i) > \delta$, we shall define the desired function g. For this, we inductively choose $g_0, g_1, \ldots, g_n$ elements of F such that:

$f = g_0$,

if $k \in \mathbf{N}$, $1 \leq k < n$ and $g_0, g_1, \ldots, g_k$ have been chosen satisfying the property:

for every $1 \leq l \leq k$ and $i = 1, 2, \ldots, l-1$,

$|g_l(m_i) - g_{l-1}(m_i)| \leq \epsilon_l$ and

for $i \in \{l+1, \ldots, n\} \cap I$, we have that $g_l(m_i) > \delta$

and $g_l(m_l) > \delta$ if $l \in I$ or $|g_l(m_l)| < \epsilon_l$ otherwise.

To choose g_{k+1} we distinguish two cases.

Case 1: $k + 1 \in I$. Then we set $g_{k+1} = g_k$.

CASE 2: $k+1 \notin I$. Then we choose g_{k+1} such that $|g_{k+1}(m_i) - g_k(m_i)| \leq \epsilon_{k+1}$ for $i \leq k$, $g_{k+1}(m_i) > \delta$ for every $i \in \{k+2, \dots, n\} \cap I$ and $|g_{k+1}(m_{k+1})| < \epsilon_{k+1}$. The existence of such a g_{k+1} follows from the properties of the set M.

This completes the inductive definition of $g_0, \dots, g_n$. It is easy to see that the final function g_n is the desired g. The proof of the lemma is complete.

1.3. THEOREM: *Every normalized weakly null sequence $(x_n)_{n \in \mathbf{N}}$ in a Banach space X has a convexly unconditional subsequence.*

Proof: Assume, by passing to a subsequence if it is needed, that $(x_n)_{n \in \mathbf{N}}$ is Schauder basic with basis constant $D \geq 1$. We inductively apply Lemma 1.2 to choose a decreasing sequence $(M_n)_{n \in \mathbf{N}}$ such that M_n satisfies the conclusion of the Lemma for $F = \{(x^*(x_n))_{n \in \mathbf{N}}, \|x^*\| \leq 1\}$, $\delta = 1/n$, $\epsilon = 1/n^3$.

We select a strictly increasing sequence $M = (m_n)_{n \in \mathbf{N}}$ such that $m_n \in M_n$.

CLAIM: *The sequence $(x_n)_{n \in M}$ is convexly unconditional.*

Indeed, given an absolutely convex combination $x = \sum_{n \in M} a_n x_n$ with $\|x\| > 1/k$ and $(\varepsilon_n)_{n \in M}$ a sequence of signs we choose $x^* \in B_{X^*}$ with $x^*(x) > 1/k$. There exists a finite $J \subset M$ such that $x^*(\sum_{n \in J} a_n x_n) > 1/k$. We set

$$J_1 = \{n \in J \colon |x^*(x_n)| > 1/2k\}.$$

Then we have $x^*(\sum_{n \in J \setminus J_1} a_n x_n) \leq 1/2k$ and hence $x^*(\sum_{n \in J_1} a_n x_n) > 1/2k$. By splitting the set J_1 into four sets, in the obvious way, we find a subset $I \subset J_1$ such that:

$|\sum_{n \in I} \varepsilon_n a_n| > 1/8k$,

$\{\varepsilon_n a_n \colon n \in I\}$ are either all non-negative or all negative and

$\{x^*(x_n) \colon n \in I\}$ are of the same sign.

We consider x^* if the sign of $x^*(x_n)$ is positive and $-x^*$ if the sign is negative and this we again denote by x^*. Thus we also have $x^*(x_n) > 1/2k$ for $n \in I$. For every $r \in \mathbf{N}$ we denote by $B(r)$ the unconditional constant of $\{x_{m_1}, \dots, x_{m_r}\}$. This means that for $G \subset \{1, \dots, r\}$,

$$\left\| \sum_{i \in G} b_i x_{m_i} \right\| \leq B(r) \cdot \left\| \sum_{i=1}^{r} b_i x_{m_i} \right\|.$$

(This happens because the norm $\|\cdot\|$ in the space of dimension r that is generated by $x_{m_1}, \dots, x_{m_r}$ is equivalent to the maximum norm with respect to this basis.)

We split I into two sets $I_1 = I \cap \{m_1, \dots, m_{2k-1}\}$ and $I_2 = I \setminus I_1$. We have

$$\left| \sum_{n \in I_1} \varepsilon_n a_n \right| > \frac{1}{16k} \text{ or } \left| \sum_{n \in I_2} \varepsilon_n a_n \right| > \frac{1}{16k}.$$

If the first condition holds, then

$$\begin{aligned}(1)\qquad \Big\|\sum_{n\in M}\varepsilon_n a_n x_n\Big\| &\geq \frac{1}{D}\Big\|\sum_{i=1}^{2k-1}\varepsilon_{m_i}a_{m_i}x_{m_i}\Big\| \geq \frac{1}{D\cdot B(2k-1)}\Big\|\sum_{n\in I_1}\varepsilon_n a_n x_n\Big\|\\ &> \frac{1}{D\cdot B(2k-1)}\Big|\sum_{n\in I_1}\varepsilon_n a_n\Big|\cdot\frac{1}{2k} > \frac{1}{D\cdot B(2k-1)}\cdot\frac{1}{32k}.\end{aligned}$$

In the second case, there exists $y^*\in B_{X^*}$ such that

(i) $\min_{I_2}\{y^*(x_n)\} > \left(1-\frac{1}{(4k)^3}\right)\cdot\frac{1}{2k}$ and

(ii) $\max\{|y^*(x_n)|: n\in\{m_{2k},\dots,m_l\}\setminus I_2\} < 1/2k(4k)^3$, where $m_l=\max I$.

Therefore

$$\begin{aligned}\Big|y^*\Big(\sum_{i=2k}^{l}\varepsilon_{m_i}a_{m_i}x_{m_i}\Big)\Big| &> \Big|y^*\Big(\sum_{n\in I_2}\varepsilon_n a_n x_n\Big)\Big| - \Big|y^*\Big(\sum_{n\in\{m_{2k},\dots,m_l\}\setminus I_2}\varepsilon_n a_n x_n\Big)\Big|\\ &> \frac{1}{16k}\left(1-\frac{1}{(4k)^3}\right)\frac{1}{2k}-\frac{1}{2k}\frac{1}{(4k)^3}\\ &= \frac{1}{32k^2}\left(1-\frac{1}{(4k)^3}-\frac{1}{(2k)^2}\right) > \frac{1}{64k^2}\end{aligned}$$

Finally

$$(2)\qquad \Big\|\sum_{n\in M}\varepsilon_n a_n x_n\Big\| > \frac{1}{2D}\Big\|\sum_{i=2k}^{l}\varepsilon_{m_i}a_{m_i}x_{m_i}\Big\| > \frac{1}{D\cdot 128k^2}.$$

From (1) and (2) we get that

$$C\left(\frac{1}{k}\right) \geq \min\left\{\frac{1}{D\cdot B(2k-1)}\cdot\frac{1}{32k},\frac{1}{D\cdot 128k^2}\right\}.$$

($C(\cdot)$ is defined in the definition of convexly unconditional.)

1.4. COROLLARY: *Let $(x_n)_{n\in\mathbf{N}}$ be a normalized weakly null sequence. Then for every $M\in[\mathbf{N}]$ there exists $L\in[M]$, $L=(l_j)_{j\in\mathbf{N}}$ such that the following property is satisfied:*

For every $k>0$ there exists $C(k)>0$ such that for every $x=\sum_{j=1}^\infty a_j x_{l_j}$ with $\|x\|=1$ and $\sum_{j=1}^\infty |a_j|<k$, then for every sequence $(\varepsilon_j)_{j\in\mathbf{N}}$ of signs we have $C(k)<\left\|\sum_{j=1}^\infty \varepsilon_j a_j x_{l_j}\right\|\leq k$.

Proof: Set $d=\sum_{j=1}^\infty |a_j|$ and $b_j=a_j/d$. Then $\sum_{j=1}^\infty b_j x_{l_j}$ is an absolutely convex combination and $\|\sum b_j x_{l_j}\|>1/d>1/k$. Hence by Theorem 1.3 we get the left inequality with $C(k) = C(1/k)$. The right is immediate from the triangle inequality.

The following consequence of Lemma 1.2 has been proved by H. Rosenthal with the use of transfinite induction.

1.5. THEOREM: *Let K be a compact space and $(f_n)_{n\in\mathbf{N}}$ a sequence of continuous characteristic functions converging pointwise to zero. Then there exists $L \in [\mathbf{N}]$, $L = (l_j)_{j\in\mathbf{N}}$ such that $(f_{l_j})_{j\in\mathbf{N}}$ is an unconditional basic sequence.*

Proof: Define $F\colon K \longrightarrow c_0(\mathbf{N})$ by the rule $F(x) = (f_n(x))_{n\in\mathbf{N}}$. Then for each $x \in K$, $F(x)$ is a finite subset of $\mathbf{N}$ and $F[K]$ is weakly compact. By Lemma 1.2, there exists $L \in [\mathbf{N}]$ such that for every $x \in K$, $G \subset F(x) \cap L$ there exists $y \in K$ such that $F(y) \cap L = G$. It is easy to check that the sequence $(f_n)_{n\in L}$ is an unconditional sequence.

1.6. COROLLARY: *Let $\mathcal{M}$ be a compact family, in the pointwise convergence topology, of finite subsets of $\mathbf{N}$. Then for every infinite subset N of $\mathbf{N}$ there exists an infinite subset M of N such that the compact family*

$$\mathcal{M}[M] = \{G \cap M\colon G \in \mathcal{M}\}$$

is adequate (i.e. for every $G \in \mathcal{M}$ and $F \subset G \cap M$ there exists $G' \in \mathcal{M}$ such that $F = G' \cap M$).

Proof: Consider the set $K = \{\chi_G\colon G \in \mathcal{M}\}$. Then K is a weakly compact subset of $c_0(\mathbf{N})$. Applying Lemma 1.2 for the sets K, N and the numbers $\delta = 1/2$, $\epsilon = 1/2$ we get an infinite set $M = (m_i)_{i\in\mathbf{N}}$ satisfying the conclusion of it. We claim that M is the desired set. Indeed, for a nonempty subset $F = (m_i)_{i\in I}$ of $G \cap M$ and for every $n \in \mathbf{N}$ such that $I \subset \{1, 2, \ldots, n\}$ we set G_n the elements of $\mathcal{M}$ such that

$$G_n \cap \{m_i\}_{i=1}^{n} = F.$$

This follows from the properties of the set M.

Set G' any cluster point of the sequence $\{G_n\}$. Then, clearly, $G' \cap \{m_i\}_{i\in\mathbf{N}} = F$ and the proof is complete.

2. Summability methods

The second section is divided into five subsections. In the first of them we study the initial properties of the two hierarchies (the Schreier and the RA). In the second we introduce the strong Cantor–Bendixson index. This is a variation of the classical C-B index and we use it in order to develop a criterion for the embedding of the $\mathcal{F}_\xi^N$ into an adequate family $\mathcal{F}$. In the third subsection we prove the basic ingredient for our main result. This is Proposition 2.3.2 on the structure of large families. Its proof is a combination of transinfinite induction with repeated use of infinite Ramsey Theory. In the fourth subsection we present the main results of the second section as mentioned in the introduction. Finally in the fifth subsection we study the anti-uniform convergence index, introduced here, which is a criterion for the existence of ℓ_ξ^1 spreading models for weakly null sequences.

THE SCHREIER HIERARCHY, THE RA HIERARCHY. *Notation:* We denote by $S_{\ell^1}^+$ the positive part of the unit sphere of $\ell^1(\mathbf{N})$. For $A = (a_n)_{n\in\mathbf{N}}$ in $S_{\ell^1}^+$ and $F = (x_n)_{n\in\mathbf{N}}$ a bounded sequence in a Banach space X, we denote by $A \cdot F$ the usual matrix product, that is:

$$A \cdot F = \sum_{n=1}^{\infty} a_n x_n.$$

2.1.1. Definition: For an infinite subset M of $\mathbf{N}$ an M **summability method** is a block sequence $(A_n)_{n\in\mathbf{N}}$ with $A_n \in S_{\ell^1}^+$ and $M = \bigcup_{n=1}^{\infty} \operatorname{supp} A_n$ where $\operatorname{supp} A_n = \{n \in \mathbf{N}: a_n \neq 0\}$.

2.1.2. Definition: Suppose that $(A_n)_{n\in\mathbf{N}}$ is an M summability method. A bounded sequence $F = (x_n)_{n\in\mathbf{N}}$ is said to be $M - (A_n)_{n\in\mathbf{N}}$ **summable** if the sequence $(A_n \cdot F)_{n\in\mathbf{N}}$ is Cesaro summable. This means that the sequence $z_n = (\sum_{k=1}^{n} A_k \cdot F)/n$ is norm convergent.

2.1.3. Remark: To each $M \in [\mathbf{N}]$, $M = (m_n)_{n\in\mathbf{N}}$, we assign the M-summability method $A_n = \{e_{m_n}\}$. Then the $M - (A_n)_{n\in\mathbf{N}}$ summability of $(x_n)_{n\in\mathbf{N}}$ is exactly the usual Cesaro summability of the subsequence $(x_n)_{n\in M}$.

DEFINITION OF THE SCHREIER HIERARCHY. Next we recall the definition of the generalized Schreier families $(\mathcal{F}_\xi)_{\xi<\omega_1}$. These are defined inductively in the following manner.

Notation: For F_1, F_2 in $[\mathbf{N}]^{<\omega}$ with $F_1 \neq \emptyset$, $F_2 \neq \emptyset$ we denote by $F_1 < F_2$ the relation $\max F_1 < \min F_2$.

We set $\mathcal{F}_0 = \{\{n\}: n \in \mathbf{N}\} \cup \{\emptyset\}$.

Suppose that $\xi = \zeta + 1$ and $\mathcal{F}_\zeta$ has been defined. We set

$$\mathcal{F}_\xi = \left\{ F \in [\mathbf{N}]^{<\omega}: F = \bigcup_{i=1}^{n} F_i, F_i \in \mathcal{F}_\zeta, n \leq F_1 < \cdots < F_n \right\} \cup \{\emptyset\}.$$

If ξ is a limit ordinal and $\mathcal{F}_\zeta$ has been defined for all $\zeta < \xi$ then we fix a strictly increasing family of non-limit ordinals $(\xi_n)_{n\in\mathbf{N}}$ with $\sup \xi_n = \xi$ and we define

$$\mathcal{F}_\xi = \left\{F \in [\mathbf{N}]^{<\omega} \colon n \le \min F \text{ and } F \in \mathcal{F}_{\xi_n} \text{ for some } n \in \mathbf{N}\right\} \cup \{\emptyset\}.$$

Remark: The use of a sequence of non-limit ordinals $(\xi_n)_{n\in\mathbf{N}}$ in the definition of $\mathcal{F}_\xi$, ξ limit, is necessary. We make this assumption in order to prove the Approximation Lemma given below.

DEFINITION OF THE RA HIERARCHY. To each $M \in [\mathbf{N}]$ and $\xi < \omega_1$ we will assign inductively an M summability method $(\boldsymbol{\xi}_n^M)_{n\in\mathbf{N}}$ in the following manner:

(i) For $\xi = 0, M = (m_n)_{n\in\mathbf{N}}$ we set $\boldsymbol{\xi}_n^M = \{e_{m_n}\}$.

(ii) If $\xi = \zeta + 1$, $M \in [\mathbf{N}]$ and $(\boldsymbol{\zeta}_n^M)_{n\in\mathbf{N}}$ has been defined then we inductively define $(\boldsymbol{\xi}_n^M)_{n\in\mathbf{N}}$ as follows. We set $k_1 = 0$, $s_1 = \min\operatorname{supp}\boldsymbol{\zeta}_1^M$, and

$$\boldsymbol{\xi}_1^M = \frac{\boldsymbol{\zeta}_1^M + \cdots + \boldsymbol{\zeta}_{s_1}^M}{s_1}.$$

Suppose that for $j = 1, 2, \ldots, n-1$, k_j and s_j have been defined and

$$\boldsymbol{\xi}_j^M = \frac{\boldsymbol{\zeta}_{k_j+1}^M + \cdots + \boldsymbol{\zeta}_{k_j+s_j}^M}{s_j}.$$

Then we set

$$k_n = k_{n-1} + s_{n-1}, \quad s_n = \min\operatorname{supp}\boldsymbol{\zeta}_{k_n+1}^M \quad \text{and}$$
$$\boldsymbol{\xi}_n^M = \frac{\boldsymbol{\zeta}_{k_n+1}^M + \cdots + \boldsymbol{\zeta}_{k_n+s_n}^M}{s_n}.$$

This completes the definition for successor ordinals.

(iii) If ξ is a limit ordinal and if we suppose that for every $\zeta < \xi$ and $M \in [\mathbf{N}]$ the sequence $(\boldsymbol{\zeta}_n^M)_{n\in\mathbf{N}}$ has been defined, then we define $(\boldsymbol{\xi}_n^M)_{n\in\mathbf{N}}$ as follows:

We denote by $(\xi_n)_{n\in\mathbf{N}}$ the strictly increasing sequence of ordinals with $\sup\xi_n = \xi$ that defines the family $\mathcal{F}_\xi$.

For $M = (m_k)_{k\in\mathbf{N}}$ we inductively define $M_1 = M$, $n_1 = m_1$,

$$M_2 = \left\{m_k \colon m_k \notin \operatorname{supp}[\boldsymbol{\xi}_{n_1}]_1^{M_1}\right\}, \quad n_2 = \min M_2,$$
$$M_3 = \left\{m_k \colon m_k \notin \operatorname{supp}[\boldsymbol{\xi}_{n_2}]_1^{M_2}\right\}, \quad n_3 = \min M_3, \qquad \text{and so on.}$$

We set

$$\boldsymbol{\xi}_1^M = [\boldsymbol{\xi}_{n_1}]_1^{M_1}, \boldsymbol{\xi}_2^M = [\boldsymbol{\xi}_{n_2}]_1^{M_2}, \ldots, \boldsymbol{\xi}_k^M = [\boldsymbol{\xi}_{n_k}]_1^{M_k}, \ldots.$$

Hence $(\boldsymbol{\xi}_n^M)_{n\in\mathbf{N}}$ has been defined. This completes the definition of RA Hierarchy.

PROPERTIES OF THE TWO HIERARCHIES. The following properties can be established inductively.

P.1: For $\xi < \omega_1$ and $M \in [\mathbf{N}]$ $(\boldsymbol{\xi}_n^M)_{n\in\mathbf{N}}$ is an M-summability method, i.e., $(\boldsymbol{\xi}_n^M)_{n\in\mathbf{N}}$ is a block sequence of elements of $S_{\ell^1}^+$ and $M = \bigcup_{n=1}^\infty \operatorname{supp} \boldsymbol{\xi}_n^M$.

P.2: For every $\xi < \omega_1$, $M \in [\mathbf{N}]$ and $n \in \mathbf{N}$ $\operatorname{supp} \boldsymbol{\xi}_n^M \in \mathcal{F}_\xi$.

P.3: For every $\xi < \omega_1$ and every $N, M \in [\mathbf{N}]$ such that $\operatorname{supp} \boldsymbol{\xi}_i^M = \operatorname{supp} \boldsymbol{\xi}_i^N$ for $i = 1, 2, \ldots, k$ we have $\boldsymbol{\xi}_i^M = \boldsymbol{\xi}_i^N$ for $i = 1, 2, \ldots, k$.

P.4: For every $M \in [\mathbf{N}]$ and $(n_k)_{k\in\mathbf{N}} \in [\mathbf{N}]$ if $M' = \bigcup_{k=1}^\infty \operatorname{supp} \boldsymbol{\xi}_{n_k}^M$ then $\boldsymbol{\xi}_k^{M'} = \boldsymbol{\xi}_{n_k}^M$.

Remark: Properties P.3 and P.4 are important for our proofs and they indicate a strong stability of the methods $(\boldsymbol{\xi}_n^M)_{n\in\mathbf{N}}$.

2.1.4. Definition: A family $\mathcal{F}$ of finite subsets of $\mathbf{N}$ is said to be **adequate** if $\mathcal{F}$ is compact and for every $F \in \mathcal{F}$, if $G \subset F$ then $G \in \mathcal{F}$.

2.1.5. Notation: If $\mathcal{F}$ is an adequate family and $L \in [\mathbf{N}]$ we denote by $\mathcal{F}[L]$ the restriction of $\mathcal{F}$ on L, that is $\mathcal{F}[L] = \mathcal{F} \cap [L]^{<\omega}$ $(= \{F \in \mathcal{F}\colon F \subset L\})$.

Clearly $\mathcal{F}[L]$ is an adequate subfamily of $\mathcal{F}$.

2.1.6. Notation: For an ordinal $\xi < \omega_1$ and $M \in [\mathbf{N}]$, $M = (m_i)_{i\in\mathbf{N}}$ we define

$$\mathcal{F}_\xi^M = \{G\colon G = (m_i)_{i\in F}, F \in \mathcal{F}_\xi\}.$$

It is easy to see that $\mathcal{F}_\xi^M$ is an adequate family.

2.1.7. Remark: It is proved readily by induction that $\mathcal{F}_\xi^M$ is a subfamily of $\mathcal{F}_\xi[M]$; on the other hand, it is not true that $\mathcal{F}_\xi[M]$ is contained in $\mathcal{F}_\xi^M$. We will show that by going to a subset N of M, $\mathcal{F}_\xi^M[\mathbf{N}]$ and $\mathcal{F}_\xi[\mathbf{N}]$ are in a sense comparable.

2.1.8. LEMMA:

(a) *For every $\zeta < \xi < \omega_1$ there exists $n \equiv n(\zeta, \xi) \in \mathbf{N}$ such that if $n \le F \in \mathcal{F}_\zeta$ then $F \in \mathcal{F}_\xi$. The same holds for $\mathcal{F}_\zeta^M, \mathcal{F}_\xi^M$.*

(b) *For every $\xi < \omega_1$, whenever $F = \{n_1 < \cdots < n_k\} \in \mathcal{F}_\xi$ and $m_i \ge n_i$ for $i \le k$ then we have $\{m_1, \ldots, m_k\} \in \mathcal{F}_\xi$. The same holds for $\mathcal{F}_\xi^M$.*

Proof: The proof of this lemma is obtained easily by induction.

2.1.9. LEMMA: *For every $M \in [\mathbf{N}]$ there exists $L \in [M], L = (l_i)_{i\in\mathbf{N}}$ satisfying the following property. For every $\xi < \omega_1$, if $A = \{l_{k_1} < \cdots < l_{k_n}\}$ is a subset of L and $A \in \mathcal{F}_\xi$ then $A \setminus \{l_{k_1}\} \in \mathcal{F}_\xi^M$.*

Proof: Let $M = (m_i)_{i\in\mathbf{N}}$. We set $l_1 = m_1$ and $l_{k+1} = m_{l_k}$ for $k \geq 2$. Let $A = \{l_{k_1} < \cdots < l_{k_n}\} \subseteq L$ with $A \in \mathcal{F}_\xi$. Then clearly we have

$$\{m_{l_{k_1}} < \cdots < m_{l_{k_n}}\} = \{l_{k_1+1} < \cdots < l_{k_n+1}\} \in \mathcal{F}_\xi^M.$$

Since we have $l_{k_1+1} < l_{k_2}, l_{k_2+1} < l_{k_3}, \ldots l_{k_{n-1}+1} < l_{k_n}$ and since the class $\mathcal{F}_\xi^M$ is spreading (i.e. satisfies (b) of Lemma 2.18), we get that $\{l_{k_2} < \cdots < l_{k_n}\} = A \setminus \{l_{k_1}\} \in \mathcal{F}_\xi^M$.

Note: The above lemma was suggested to us by G. Androulakis and E. Odell as an alternative of a similar result we had which was weaker and much more complicated. We wish to thank them for their permission to include here their argument that is also contained in [An–Od].

2.1.10 PROPOSITION: *For every $M \in [\mathbf{N}]$, $\epsilon > 0$ there exists $L \in [M]$ such that: For every $1 \leq \xi < \omega_1$, $P \in [L]$, $n \in \mathbf{N}$ there exists $G \in \mathcal{F}_\xi^M$ such that $\langle \boldsymbol{\xi}_n^P, G\rangle > 1-\epsilon$.*

Proof: Take L given by the Lemma 2.19 which works for the set $\{m \in M: m > 1/\epsilon\}$. If $F = \operatorname{supp} \boldsymbol{\xi}_n^P$, then it is easily verified that the set $G = F \setminus \min F$ satisfies our assertion.

2.1.11. COROLLARY: *For every $M \in [\mathbf{N}]$there exists $L \in [M]$ such that: For every $1 \leq \xi < \omega_1$, $P \in [L]$, $n \in \mathbf{N}$ there exists $G \in \mathcal{F}_\xi^M$ such that*

$$\langle \boldsymbol{\xi}_n^P, G\rangle > \tfrac{1}{2}.$$

Proof: We apply the previous Proposition for $\epsilon = \frac{1}{2}$.

2.1.12. Remark: A consequence of the above Lemma is that the Schreier hierarchy is in a sense universal.

Indeed, consider $f: \mathbf{N} \longrightarrow \mathbf{N}$ any strictly increasing function and define

$$\mathcal{F}_1^f = \{F \subset \mathbf{N}: \min F = n, \#F \leq f(n)\}.$$

$\mathcal{F}_1^f$ is an adequate family and the regular $\mathcal{F}_1$ is $\mathcal{F}_1^f$ for $f = \mathrm{id}_{\mathbf{N}}$.

By iteration we produce $(\mathcal{F}_\xi^f)_{\xi<\omega_1}$ and the repeated averages hierarchy $[\boldsymbol{\xi}^f]_n^M$ for $M \in [\mathbf{N}]$, $n \in \mathbf{N}$.

Next define the set $M = (m_i)_{i\in\mathbf{N}}$ by the rule $m_i = f(i)$. Then observe that if

$$\mathcal{F}_\xi^{f,M} = \{(m_i)_{i\in F}, F \in \mathcal{F}_\xi^f\},$$

we have that

$$\mathcal{F}_\xi^{f,M} = \mathcal{F}_\xi[M].$$

Therefore from the above corollary we get that there exists $L \in [M]$ such that for every $P \in [L]$, $n \in \mathbf{N}$ there exists $G \in \mathcal{F}_\xi^M$ with $\langle [\boldsymbol{\xi}^{f,M}]_n^P, G\rangle > \frac{1}{2}$.

This shows that $\mathcal{F}_\xi^{f,M}$, $\mathcal{F}_\xi^M$ are comparable on the set L.

2.1.13. Lemma (Approximation Lemma): *Let $\xi < \omega_1$, $M \in [\mathbf{N}]$, $\epsilon > 0$. We set $W = \mathrm{co}(\{\xi_n^N : n \in \mathbf{N}, N \in [M]\})$. Then for every ordinal ζ such that $\xi \leq \zeta < \omega_1$, $L \in [M]$ there exists $L_\zeta \in [L]$ satisfying the following property:*

For every $L' \in [L_\zeta]$, $n \in \mathbf{N}$ we have that

$$d_{\ell^1}(\zeta_n^{L'}, W) < \epsilon.$$

Proof: Fix $\xi < \omega_1$ and $M \in [\mathbf{N}]$. We shall prove the Lemma by induction for ζ greater than ξ, every $L \in [M]$ and $\epsilon > 0$.

(i) $\zeta = \eta + 1$. Indeed, if $M \in [\mathbf{N}]$ and $\epsilon > 0$ then there exist L_η satisfying the conclusion for the ordinal η. Set $L_\zeta = L_\eta$. It is obvious that for every $L' \in [L_\zeta]$, $n \in \mathbf{N}$ we get the desired property $d_{\ell^1}(\zeta_n^{L'}, W) < \epsilon$.

(ii) ζ is a limit ordinal. Then fix the strictly increasing sequence $(\zeta_n)_{n\in\mathbf{N}}$ of successor ordinals such that $\sup \zeta_n = \zeta$ and $(\zeta_n)_{n\in\mathbf{N}}$ defines the family $\mathcal{F}_\zeta$. Since each ζ_n is a successor ordinal it has the form $\zeta_n = \xi_n + 1$.

Choose $L_0 \in [M]$ with $\min L_0 = m_1$ and $1/m_1 < \epsilon/4$. We inductively choose $L_0 \supset L_1 \supset \cdots \supset L_k \supset \cdots$ such that if $n_k = \min L_k$ then $(n_k)_{k\in\mathbf{N}}$ is strictly increasing and $L_k = L_{n_{k-1}}$ for $M = L_{k-1}$, $\zeta = \xi_{n_{k-1}}$, $\epsilon/2$.

Claim: *The set $N = (n_k)_{k\in\mathbf{N}}$ is the desired L_ξ.*

Indeed, let $L' \in [L_\xi]$, $n \in \mathbf{N}$. Then, by definition, $\boldsymbol{\zeta}_k^{L'} = [\boldsymbol{\zeta}_{n_k}]_1^{L'_k}$, where $n_k = \min\operatorname{supp}\boldsymbol{\zeta}_n^{L'}$, $L'_k = \{m \in L' : n_k \leq m\}$. It is clear that $L'_k \setminus \{n_k\} \subset L_k$. Since $\zeta_{n_k} = \xi_{n_k} + 1$, again by definition, $[\boldsymbol{\zeta}_{n_k}]_1^{L'_k}$ is an average of n_k many successive elements of $\left([\boldsymbol{\xi}_{n_k}]_n^{L'_k}\right)_{n\in\mathbf{N}}$. Since all of them except the first one are $\epsilon/2$ approximated by convex combination of W and $1/n_k < \epsilon/4$ we get that $d_{\ell^1}([\boldsymbol{\zeta}_{n_k}]_1^{L'_k}, W) < \epsilon$ and hence $d_{\ell^1}(\boldsymbol{\zeta}_k^{L'}, W) < \epsilon$. This completes the proof of the lemma.

Strong Cantor–Bendixson index.

2.2.1 Definition: Let $\mathcal{F}$ be an adequate family. For $L \in [\mathbf{N}]$ we define the strong Cantor–Bendixson derivative of $\mathcal{F}[L]$ by the rule:

$$\mathcal{F}[L]^{(1)} = \{A \in \mathcal{F}[L] : \forall N \in [L], A \text{ is a cluster point of } \mathcal{F}[A \cup N]\}.$$

2.2.2 Remarks: (1) It is clear that $\mathcal{F}[L]^{(1)}$ is a closed and nowhere dense subset of $\mathcal{F}[L]$ which is also adequate; in fact we have

$$\mathcal{F}[L]^{(1)} = \bigcup_{A\in[L]^{<\omega}} \bigcap_{N\in[L]} (\mathcal{F}[A\cup N])',$$

where $(\mathcal{F}[A\cup N])'$ denotes the "usual" Cantor–Bendixson derivative of $\mathcal{F}[A\cup N]$.

(2) It is also easily verified that $\mathcal{F}[L]^{(1)} \neq \emptyset$ iff $\mathcal{F}[N]$ is an infinite set for every $N \in [L]$ (iff $\mathcal{F}[N]^{(1)} \neq \emptyset$ for every $N \in [L]$).

If $\xi = \zeta + 1$ then we inductively define $\mathcal{F}[L]^{(\xi)} = (\mathcal{F}[L])^{(\zeta)})[L]^{(1)}$. If ξ is a limit ordinal then we set

$$\mathcal{F}[L]^{(\xi)} = \bigcap_{\zeta<\xi} \mathcal{F}[L]^{(\zeta)}.$$

We define the **S.C.B. index of** $\mathcal{F}[L]$ as the smallest ordinal ξ_0 such that $\mathcal{F}[L]^{(\xi_0)} = \emptyset$.

We denote this index by $s(\mathcal{F}[L])$. Note that $s(\mathcal{F}[L])$ is a successor ordinal.

The following Lemma is easily proved by induction on ξ.

2.2.2.a. LEMMA: *For every $N \in [L]$, $\xi < \omega_1$,*

$$\mathcal{F}[L]^{(\xi)} \cap [N]^{<\omega} \subseteq \mathcal{F}[N]^{(\xi)}.$$

2.2.3 PROPOSITION: *If $\xi < \omega_1$ and $L \in [\mathbf{N}]$ is such that $s(\mathcal{F}[L]) > \xi$, then for every $N \in [L]$ we have that $s(\mathcal{F}[N]) > \xi$.*

Proof: We will show that for every ordinal ζ satisfying $\zeta < \xi$ and for each $N \in [L]$ we have that $s(\mathcal{F}[N]) > \zeta + 1$.

It follows from Remark 2.2.2 (2) above that it is enough to prove that $\mathcal{F}[N]^{(\zeta)}[N_1]$ is infinite for all $N_1 \in [N]$. By this same remark, we know that $\mathcal{F}[L]^{\zeta)}[N_1]$ is infinite. By Lemma 2.2.2.a, $\mathcal{F}[L]^{(\zeta)}[N] \subseteq \mathcal{F}[N]^{(\zeta)}$ and so $\mathcal{F}[L]^{(\zeta)}[N_1] = \mathcal{F}[L]^{(\zeta)}[N] \cap [N_1]^{<\omega} \subseteq \mathcal{F}[N]^{(\zeta)} \cap [N_1]^{<\omega} = \mathcal{F}[N]^{(\zeta)}[N_1]$. Thus $\mathcal{F}[N]^{(\zeta)}[N_1]$ is infinite as required. This completes the proof.

For N, L subsets of $\mathbf{N}$ we say that N is almost contained in L if the difference $L \setminus N$ is a finite set.

2.2.4 PROPOSITION: *If $\mathcal{F}$ is an adequate family, $\xi < \omega_1$, $N, L \in [\mathbf{N}]$, such that N is almost contained in L, then $s(\mathcal{F}[L]) > \xi$ implies that $s(\mathcal{F}[N]) > \xi$.*

Proof: Similar to the previous one.

2.2.5 Remark: It is easily verified by induction on ξ that for every $\xi < \omega_1$ we have $s(\mathcal{F}_\xi) = \omega^\xi + 1$ (cf. also Prop. 4.10 in [Al–Ar]).

Notation: In the sequel we will denote by $\mathcal{F}[L]^{(\xi)}$ the ξ-derivative of $\mathcal{F}[L]$, while for $N \in [L]$ we denote by $\mathcal{F}^{(\xi)}[N]$ the restriction $\mathcal{F}[L]^{(\xi)}[N]$ of $\mathcal{F}[L]^{(\xi)}$ on the set N. Notice that $\mathcal{F}^{(\xi)}[N] = \mathcal{F}[L]^{(\xi)} \cap [N]^{<\omega} \subset \mathcal{F}[N]^{(\xi)}$ by Lemma 2.2.2.a.

2.2.6. THEOREM: *Let $\mathcal{F}$ be an adequate family. If $L \in [\mathbf{N}]$ such that $s(\mathcal{F}[L]) > \omega^\xi$ then there exists $M \in [L]$, $M = (m_i)_{i\in\mathbf{N}}$ satisfying the property: $\mathcal{F}_\xi^M$ is a subfamily of $\mathcal{F}[M]$.*

In order to prove this theorem we shall use a method developed by Kiriakouli–Negrepontis [M–N]. This method consists of a double induction. We start with the next definition.

Definition: An n-tuple of ordinals $< \omega_1$ $(\xi_1, \dots, \xi_n)$ has property (A) if for every adequate family $\mathcal{F}$ with

$$s(\mathcal{F}) > \omega^{\xi_n} + \cdots + \omega^{\xi_1}$$

and for every $L \in [\mathbf{N}]$ there exists $N \in [L]$, $N = (n_i)_{i\in\mathbf{N}}$ such that for every

$$F_1 \in \mathcal{F}_{\xi_1}, \dots, F_n \in \mathcal{F}_{\xi_n} \quad \text{with } F_1 < F_2 < \cdots < F_n$$

the set $\{n_i : i \in \bigcup_{k=1}^n F_k\}$ belongs to $\mathcal{F}[N]$.

2.2.7 LEMMA: *Suppose that $(\xi_1, \xi_2, \dots, \xi_n)$ has property (A) and $\zeta < \omega_1$. Then $(\zeta, \xi_1, \xi_2, \dots, \xi_n)$ also has property (A).*

Proof: We proceed by induction on $\zeta < \omega_1$.

CASE 1: $\zeta = 0$. Given $(\eta_1, \dots, \eta_k)$ with property (A) we show that $(0, \eta_1, \dots, \eta_k)$ also has property (A).

Indeed, start with $\mathcal{F}$ adequate such that

$$s(\mathcal{F}) > \omega^{\eta_k} + \cdots + \omega^{\eta_1} + 1.$$

Set $\zeta = \omega^{\eta_k} + \cdots + \omega^{\eta_1}$. Since $s(\mathcal{F}) > \zeta + 1$ we get from Proposition 2.2.3 that for every $L \in [\mathbf{N}]$, $s(\mathcal{F}[L]) > \zeta + 1$ and hence $\mathcal{F}[L]^{(\zeta)}$ is an infinite set. Therefore since $\mathcal{F}[L]^{(\zeta)}$ is adequate there exists $M \in [L]$, $M = (m_i)_{i\in\mathbf{N}}$ such that $\{\{m_i\} : i \in \mathbf{N}\}$ is a subfamily of $\mathcal{F}[L]^{(\zeta)}$.

Observe that the set

$$\mathcal{G}_{m_1} = \{F \in \mathcal{F}[M \setminus \{m_1\}] : \{m_1\} \cup F \in \mathcal{F}[M]\}$$

is an adequate family and $s(\mathcal{G}_{m_1}) > \zeta$. This is so since $s(\mathcal{F}[M]) > \zeta$ and $\{m_1\} \in \mathcal{F}[M]^{(\zeta)}$. From the inductive assumption there exists $M_1 \in [M]$ such that if

$$F_1 \in \mathcal{F}_{\eta_1}, \dots, F_k \in \mathcal{F}_{\eta_k} \quad \text{with } F_1 < F_2 < \cdots < F_k$$

then

$$D = \{m_i^1 \colon i \in \bigcup_{j=1}^{k} \mathcal{F}_{\eta_j}\} \in \mathcal{G}_{m_1}[M_1],$$

where $M_1 = (m_i^1)_{i \in \mathbf{N}}$.

Then clearly for any such D the set $\{m_1\} \cup D$ belongs to $\mathcal{F}[M]$.

Set $n_1 = m_1$, $n_2 = m_1^1$ and repeat the same procedure by defining $\mathcal{G}_{n_2}$ and finding $M_2 \in [M_1]$ such that $M_2 = (m_i^2)$, $n_2 < m_1^2$ and if

$$F_1 \in \mathcal{F}_{\eta_1}, \dots, F_k \in \mathcal{F}_{\eta_k} \text{ satisfying } F_1 < F_2 < \cdots < F_k$$

then the set $\{n_2\} \cup \{m_i^2 \colon i \in \bigcup_{j=1}^{k} F_j\}$ belongs to $\mathcal{F}[M_2]$. Following the same procedure, we inductively choose n_l, M_l with

$$M = M_0 \supset M_1 \supset M_2 \supset \cdots \supset M_l, \quad \text{and} \quad n_l \in M_{l-1}$$

satisfying the above properties. It now follows immediately that the set $N = (n_l)_{l \in \mathbf{N}}$ satisfies the required properties and hence $(0, \eta_1, \eta_2, \dots, \eta_k)$ has property (A).

CASE 2: $\xi = \zeta + 1$. Then by the inductive hypothesis, for every k-tuple $(\eta_1, \eta_2, \dots, \eta_k)$ with the property (A) and every $l \in \mathbf{N}$ the $l + k$-tuple $(\underbrace{\zeta, \dots, \zeta}_{l \text{ times}}, \eta_1, \dots, \eta_k)$ has the property (A).

For every $L \in [\mathbf{N}]$ such that

$$s(\mathcal{F}[L]) > \omega^{\eta_k} + \cdots + \omega^{\eta_1} + \omega^{\xi}$$

we have that for every $l \in \mathbf{N}$

$$s(\mathcal{F}[L]) > \omega^{\eta_k} + \cdots + \omega^{\eta_1} + \underbrace{\omega^{\zeta} + \cdots + \omega^{\zeta}}_{l \text{ times}}.$$

Hence we can find $L \supset L_1 \supset \cdots \supset L_l \supset \cdots$ with L_l satisfying the property:

$$\text{if } F_1 \in \mathcal{F}_{\zeta}, \dots, F_l \in \mathcal{F}_{\zeta}, F_{l+1} \in \mathcal{F}_{\eta_1}, \dots, F_{l+k} \in \mathcal{F}_{\eta_k}$$

$$\text{and } F_1 < \cdots < F_{l+k} \text{ then } \left\{ m_i^l \colon i \in \bigcup_{j=1}^{l+k} F_j \right\} \in \mathcal{F}[L_l].$$

Then if $N = (n_l)_{l\in\mathbf{N}}$ with $n_l \in L_l$, it is easy to see that N satisfies the required properties, hence $(\xi, \eta_1, \ldots, \eta_k)$ has property (A).

CASE 3: *ξ is a limit ordinal.* The proof is similar to the previous case.

Proof of the theorem: We shall use induction on ξ. The result is obvious for $\xi = 0$. Let $1 \le \xi < \omega_1$ and assume that it is true for every $\zeta < \xi$ and each adequate family $\mathcal{F}$ with $s(\mathcal{F}) > \omega^\xi$.

CASE 1: $\xi = \zeta + 1$. Since $s(\mathcal{F}) > \omega^\xi > \omega^\zeta \cdot l$ and $\underbrace{(\zeta, \zeta, \ldots, \zeta)}_{l \text{ times}}$ has property (A) then for any $L \in [\mathbf{N}]$ choose

$$L = L_0 \supset L_1 \supset L_2 \supset \cdots \supset L_l \supset \cdots$$

such that L_l witnesses property (A) for the set $\mathcal{F}[L_{l-1}]$ and the l-tuple $(\zeta, \ldots, \zeta)$.

It is easy to show that for any $N \in [L]$ such that $N = (n_l)_{l\in\mathbf{N}}$ and $n_l \in L_l$ $\mathcal{F}[N]$ satisfies the inductive assumption for the ordinal ξ.

CASE 2: *ξ is a limit ordinal.* Let $(\xi_n)_{n\in\mathbf{N}}$ be the strictly increasing sequence with $\sup \xi_n = \xi$ that defines the family $\mathcal{F}_\xi$. For every $L \in [\mathbf{N}]$ we choose $L = L_0 \supset L_1 \supset \cdots \supset L_n \supset \cdots$ such that L_n witnesses property (A) for (ξ_n) and the set $\mathcal{F}[L_{n-1}]$. If we set $N = (k_n)_{n\in\mathbf{N}}$ such that $k_n \in L_n$ then we easily check that $\mathcal{F}[N]$ satisfies the inductive assumption for the ordinal ξ. The proof of the theorem is complete.

Remark: It is clear that the following statement is a reformulation of Theorem 2.2.6: For every $\xi < \omega_1$ the 1-tuple (ξ) has property (A). This observation together with Lemma 2.2.7 immediately gives the following

2.2.8 COROLLARY: *For every $n \in \mathbf{N}$ and each $\xi_1, \ldots, \xi_n < \omega_1$ the n-tuple $(\xi_1, \ldots, \xi_n)$ has property (A).*

Remark: For $k_1, \ldots, k_n \in \mathbf{N}$ and $\xi_1, \cdots, \xi_n < \omega_1$ we denote by $((\mathcal{F}_{\xi_1})^{k_1}, \ldots, (\mathcal{F}_{\xi_n})^{k_n})$ the set of all subsets F of $\mathbf{N}$ that can be written in the form

$$F = \underbrace{F_1^1 \cup \cdots \cup F_{k_1}^1} \cup \underbrace{F_1^2 \cup \cdots \cup F_{k_2}^2} \cup \cdots \cup \underbrace{F_1^n \cup \cdots \cup F_{k_n}^n},$$

where $F_1^1 < \cdots < F_{k_n}^n$ and $F_j^i \in \mathcal{F}_{\xi_i}$ for all $i \le n$ and $j \le k_i$. These families are defined in [O–TJ–W], where it is also proved that if a family $\mathcal{F}$ is adequate and spreading and has Cantor–Bendixson index not exceeding $\omega^{\xi_1}k_1 + \cdots + \omega^{\xi_n}k_n$ then for some $M \in [\mathbf{N}]$, $M = (m_i)_{i\in\mathbf{N}}$,

$$\mathcal{F}^M \subseteq ((\mathcal{F}_{\xi_1})^{k_1}, \ldots, (\mathcal{F}_{\xi_n})^{k_n})$$

where $\mathcal{F}^M = \{\{m_i\}_{i\in F}\colon F \in \mathcal{F}\}$.

It is clear from Corollary 2.2.8 that if $\mathcal{F}$ is an adequate family with $s(\mathcal{F}) > \omega^{\xi_n}\cdot k_n + \cdots + \omega^{\xi_1}\cdot k_1$, then for every $L \in [\mathbf{N}]$ there exists $N \in [L]$, $N = (n_i)_{i\in\mathbf{N}}$ such that for every $F \in ((\mathcal{F}_{\xi_1})^{k_1}, \ldots, (\mathcal{F}_{\xi_n})^{k_n})$ the set $\{n_i\colon \in F\}$ belongs to $\mathcal{F}[N]$.

Large families.

2.3.1 Definition: Let $\mathcal{F}$ be an adequate family, $M \in [\mathbf{N}]$, $\xi < \omega_1$ and $\epsilon > 0$. We say that $\mathcal{F}$ is (M, ξ, ϵ) **large** if for every $N \in [M]$ and every $n \in \mathbf{N}$ we have that

$$\sup_{F\in\mathcal{F}} \langle \boldsymbol{\xi}_n^N, F\rangle > \epsilon.$$

2.3.2 Proposition: *If $\mathcal{F}$ is an adequate family which is (M, ξ, ϵ) large, then for every $L \in [M]$ there exists $N \in [L]$ such that $s(\mathcal{F}[N]) > \omega^\xi$.*

This proposition is one of the basic ingredients for the proof of the main Theorems of this section. This result in connection with Theorem 2.2.6 shows that every (M, ξ, ϵ) large family $\mathcal{F}$ contains a family $\mathcal{F}_\xi^{N'}$. Hence the summability methods $\{(\boldsymbol{\xi}_n^N)_{n\in\mathbf{N}}, N \in [M]\}$ are sufficiently many to describe the Schreier family $\mathcal{F}_\xi^{N'}$. The proof of the proposition depends strongly on Theorem 0.1 and the stability properties P.3–P.4 of the RA hierarchy.

Proof of the proposition: We proceed by induction. The inductive hypothesis is the statement of the proposition.

Case 1: $\xi = 0$. This is the easiest case since the result immediately follows from the definitions.

Case 2: *ξ is a limit ordinal.* In this case we prove first the following.

Claim: *For every ordinal ζ with $\zeta < \xi$ and every $L \in [M]$ there exists $N \in [L]$ such that $s(\mathcal{F}[N]) > \omega^\zeta$.*

Indeed, given $L \in [M]$ we define a partition of $[L]$ into A_1, A_2 by the rule:

$$A_1 = \left\{ N \in [L]\colon \sup_{F\in\mathcal{F}} \langle \boldsymbol{\zeta}_1^N, F\rangle \le \frac{\epsilon}{2} \right\},$$

$A_2 = [L] \setminus A_1$. Notice that if $N = (m_i)_{i\in\mathbf{N}}$ and $N' = (m'_i)_{i\in\mathbf{N}}$ are such that $m_i = m'_i$ for all $i \le k = \max\operatorname{supp}\boldsymbol{\zeta}_1^N$ then by P.3 we get that $\boldsymbol{\zeta}_1^N = \boldsymbol{\zeta}_1^{N'}$, hence A_1 is an open set. Therefore from Theorem 0.1 we get that there exists $L_1 \in [L]$ such that either $[L_1] \subset A_1$ or $[L_1] \subset A_2$.

Assume that $[L_1] \subset A_1$. Then by P.4 we have that for every $N \in [L_1]$ and every $n \in \mathbf{N}$, $\sup_{F\in\mathcal{F}}\langle \boldsymbol{\zeta}_n^N, F\rangle \le \epsilon/2$.

This is so since any such $\boldsymbol{\zeta}_n^N$ is equal to $\boldsymbol{\zeta}_1^{N'}$ for some $N' \in [N]$.

But then, from Lemma 2.1.13, there exists $L_\xi \in [L_1]$ such that for every $n \in \mathbf{N}$, $d_{\ell^1}(\boldsymbol{\xi}_n^{L_\xi}, W) < \epsilon/2$ where $W = \text{co}(\{\boldsymbol{\zeta}_n^N\colon n \in \mathbf{N}, N \in [L_1]\})$. Hence $\sup_{F\in\mathcal{F}}\langle \boldsymbol{\xi}_n^{L_\xi}, F\rangle \le \epsilon/2 + \epsilon/2 = \epsilon$, a contradiction; therefore $[L_1] \subset A_2$.

This means that the family $\boldsymbol{\mathcal{F}}[L_1]$ is $(L_1, \zeta, \epsilon/2)$ large and by the inductive assumption we get that there exists $N \in [L_1]$ such that $s(\boldsymbol{\mathcal{F}}[N]) > \omega^\zeta$. Next choose a strictly increasing sequence of ordinals $(\xi_n)_{n\in\mathbf{N}}$ with $\sup \xi_n = \xi$.

Inductively we choose $L = L_0 \supset L_1 \supset L_2 \supset \cdots \supset L_n \supset \cdots$ such that $s(\boldsymbol{\mathcal{F}}[L_n]) > \omega^{\xi_n}$.

It is easy to see that for every $N \in [L]$ such that N is almost contained in L_n we have that $s(\boldsymbol{\mathcal{F}}[N]) > \omega^{\xi_n}$, and therefore $s(\boldsymbol{\mathcal{F}}[N]) > \omega^\xi$. The proof for case 2 is complete.

CASE 3: $\xi = \zeta + 1$. We start with the following Lemma, the proof of which uses Theorem 0.1.

2.3.3. LEMMA: *Let $\xi = \zeta + 1$, $M \in [\mathbf{N}]$, $\epsilon > 0$ and $\boldsymbol{\mathcal{F}}$ an adequate family that is (M, ξ, ϵ) large. Then for every $L \in [M]$ and every $n \in \mathbf{N}$ there exists $L_n \in [L]$ such that for every $N \in [L_n]$ and $k \in \mathbf{N}$*

$$\sup_{F\in\boldsymbol{\mathcal{F}}} \min\{\langle \boldsymbol{\zeta}_k^N, F\rangle\colon 1 \le k \le n\} > \epsilon/2.$$

Proof: Consider $L \in [M]$ and $n \in \mathbf{N}$, and define a partition of $[L]$ into A_1, A_2 by the rule

$$A_1 = \{N \in [L]\colon \exists F \in \boldsymbol{\mathcal{F}}, \{\langle \boldsymbol{\zeta}_k^N, F\rangle > \epsilon/2 \text{ for } k = 1, \dots, n\}$$

and $A_2 = [L] \setminus A_1$.

As in the previous lemma, A_1 is an open set hence, by Theorem 0.1, there exists $L_n \in [L]$ such that $[L_n] \subset A_1$ or $[L_n] \subset A_2$.

We will show that the second case is not possible and this will prove the lemma.

Indeed, assuming that $[L_n] \subset A_2$ we get that for every $N \in [L_n]$ and every $k_1 < k_2 < \cdots < k_n$ and every $F \in \boldsymbol{\mathcal{F}}$

$$\text{(1)} \qquad \min\left\{\left\langle \boldsymbol{\zeta}_{k_i}^N, F\right\rangle, i = 1, 2, \dots n\right\} \le \epsilon/2.$$

This follows from the fact that there exists $N' \subset N$ such that $\boldsymbol{\zeta}_i^{N'} = \boldsymbol{\zeta}_{k_i}^N$ for all $i = 1, \dots, n$.

Each $\boldsymbol{\xi}_m^{L_n}$ is an average of successive elements of $(\boldsymbol{\zeta}_k^{L_n})_{k\in\mathbf{N}}$, that is

$$\boldsymbol{\xi}_m^{L_n} = \frac{1}{s_m}\left(\boldsymbol{\zeta}_{k_m}^{L_n} + \cdots + \boldsymbol{\zeta}_{k_m+s_m}^{L_n}\right)$$

and $(s_m)_{m\in\mathbf{N}}$ is stictly increasing. Choose $F \in \boldsymbol{\mathcal{F}}$ such that $\langle \boldsymbol{\xi}_m^{L_n}, F\rangle > \epsilon$.

Then for large s_m we get that $\#\left\{i\colon \langle \boldsymbol{\zeta}_{k_m+1}^{L_n}, F\rangle > \epsilon/2\right\} > n$. But this contradicts (1) and the proof is complete.

2.3.4 LEMMA: *Assume that $\xi, M, \epsilon, \boldsymbol{\mathcal{F}}$ are as in the previous lemma. Then for every $L \in [M]$ there exists $N \in [L]$ such that for every $n \in \mathbf{N}$ and $N' \in [N]$ with $\min N' \geq n$ we have that*

$$\sup_{F\in\boldsymbol{\mathcal{F}}} \min\{\langle \boldsymbol{\zeta}_k^{N'}, F\rangle \colon k = 1, \ldots, n\} > \epsilon/2.$$

Proof: We inductively apply the previous lemma. Choose $L \supset L_1 \supset \cdots \supset L_n \supset \cdots$ such that for all n the set L_n satisfies the conclusion of the previous lemma. Then any set $N = (m_n)_{n\in\mathbf{N}}$ with the property $m_n \in L_n$ has the desired property.

2.3.5 LEMMA: *Let $\zeta < \omega_1$, $M \in [\mathbf{N}]$, $\epsilon > 0$ and $\boldsymbol{\mathcal{F}}$ be an adequate family. Suppose that for some $n \in \mathbf{N}$ we have that for every $L \in [M]$*

$$\sup_{F\in\boldsymbol{\mathcal{F}}} \min\{\langle \boldsymbol{\zeta}_k^{L}, F\rangle \colon k = 1, \ldots, n\} > \epsilon.$$

Suppose that ζ satisfies the inductive assumption. Then for every $L \in [M]$ there exists $N \in [L]$ such that

$$s(\boldsymbol{\mathcal{F}}[N]) > \omega^{\zeta} \cdot n.$$

Proof: We proceed by induction on $\mathbf{N}$.

CASE 1: $k = 1$. As we have shown in previous proofs, the fact that for $N \in [M]$, $\sup_{F\in\boldsymbol{\mathcal{F}}}\langle \boldsymbol{\zeta}_1^{N}, F\rangle > \epsilon$, implies that $\boldsymbol{\mathcal{F}}$ is (M, ζ, ϵ) large, hence by the inductive assumption every $L \in [M]$ contains infinite subset N such that $s(\boldsymbol{\mathcal{F}}[N]) > \omega^{\zeta}$.

CASE 2: $k = n$. Assume that the Lemma has been proved for all $k = 1, 2, \ldots, n-1$.

Given $L \in [M]$, since for every $N \in [L]$ the vector $\boldsymbol{\zeta}_1^N$ has finite support and rational coefficients we get that the set $\{\boldsymbol{\zeta}_1^N \colon N \in [L]\}$ is countable and we order it as $(\boldsymbol{\zeta}_n)_{n\in\mathbf{N}}$. Consider $\boldsymbol{\zeta}_1$ and fix $L_1 \in [L]$ with

$$\max \operatorname{supp} \boldsymbol{\zeta}_1 < \min L_1.$$

Let $\{F_i\}_{i=1}^d$ be an enumeration of all nonempty subsets of $\operatorname{supp} \boldsymbol{\zeta}_1$. We define a partition of $[L_1]$ into a family $(A_i)_{i=1}^d$ defined by the rule

$A_i = \{N \in [L_1] : \text{if } N' = \operatorname{supp} \boldsymbol{\zeta}_1 \cup N \text{ and } \exists F \in \mathcal{F} \text{ satisfying}$

$$\min\{\langle \boldsymbol{\zeta}_k^{N'}, F\rangle : k = 1, \dots, n\} > \epsilon$$
$$\text{and } F \cap \operatorname{supp} \boldsymbol{\zeta}_1^{N'} = F \cap \operatorname{supp} \boldsymbol{\zeta}_1 = F_i\}.$$

Each A_i is an open set, hence by Theorem 0.1 we get that there exist i_0 and $S_1 \in [L_1]$ such that for every $N \in [S_1]$ there exists $F \in \mathcal{F}$ with:

$$\min\{\langle \boldsymbol{\zeta}_k^{N'}, F\rangle : k = 1, \dots, n\} > \epsilon$$

and

$$\operatorname{supp} \boldsymbol{\zeta}_1^{N'} \cap F = F_{i_0}.$$

Set $G_1 = F_{i_0}$ and consider the set

$$\mathcal{F}_{G_1} = \{F \in \mathcal{F} : F \cap \operatorname{supp} \boldsymbol{\zeta}_1 = G_1\}.$$

Then it is easy to see that $S_1, \zeta, \mathcal{F}_{G_1}, n-1$ satisfy the inductive assumptions, hence there exists $N_1 \in [S_1]$ such that

$$s(\mathcal{F}_{G_1}[N_1]) > \omega^\zeta \cdot (n-1).$$

As a consequence of this we get that $G_1 \in \mathcal{F}_{G_1}[N_1]^{(n-1)\cdot\omega^\zeta}$.

Inductively choose $L \supset N_1 \supset \cdots \supset N_k \supset \cdots$ and $(G_k)_{k\in\mathbf{N}}$ such that

(i) $\max \operatorname{supp} \boldsymbol{\zeta}_k < \min N_k$

(ii) $G_k \subset \operatorname{supp} \boldsymbol{\zeta}_k$, $G_k \in \mathcal{F}$ and $\langle \boldsymbol{\zeta}_k, G_k\rangle > \epsilon$,

(iii) $G_k \in \mathcal{F}_{G_k}[N_k]^{(\omega^\zeta\cdot(n-1))}$,

where $\mathcal{F}_{G_k} = \{F \in \mathcal{F} : F \cap \operatorname{supp} \boldsymbol{\zeta}_k = G_k\}$.

The choice is done as in the case $\boldsymbol{\zeta}_1$.

Choose a set N that is almost contained in N_k for all $k \in \mathbf{N}$.

CLAIM: *For every $k \in \mathbf{N}$ the set G_k belongs to $\mathcal{F}_{G_k}[N]^{(\omega^\zeta\cdot(n-1))}$, where $\mathcal{F}_{G_k}[N]$ is defined as:*

$$\mathcal{F}_{G_k}[N] = \{F \in \mathcal{F}[\operatorname{supp} \boldsymbol{\zeta}_k \cup N] : F \cap \operatorname{supp} \boldsymbol{\zeta}_k = G_k\}.$$

Indeed, set $N^k = \{m \in N : \max \operatorname{supp} \boldsymbol{\zeta}_k < m\}$.

Then since $N \setminus N^k$ is finite, we get that $G_k \in \mathcal{F}_{G_k}[N]$ provided that $G_k \in \mathcal{F}_{G_k}[N^k]$. Further, N^k is almost contained in N_k and, from (iii) and the fact that $\omega^\zeta \cdot (n-1)$ is a limit ordinal, we get that $G_k \in \mathcal{F}_{G_k}[N]$.

To finish the proof of the lemma we prove the following.

CLAIM 2: *There exists $N' \in [N]$ such that $\mathcal{F}[N']^{(\omega^\zeta \cdot n)} \neq \emptyset$.*

Indeed, consider the family

$$\mathcal{G}[N] = \{F: \exists k \in \mathbf{N} \text{ with } G_k \subset N, F \subset G_k\}.$$

It is easy to see that $\mathcal{G}[N]$ is an adequate family. Further, for $G_k \subset N$ we have $\mathcal{F}_{G_k}[N] \subset \mathcal{F}[N]$, hence for every $F \in \mathcal{G}[N]$ we have that $F \in \mathcal{F}[N]^{(\omega^\zeta \cdot (n-1))}$.

So we get that $\mathcal{G}[N] \subset \mathcal{F}[N]^{(\omega^\zeta \cdot (n-1))}$.

Notice also that for every $S \in [N]$ there exists $G_k \in \mathcal{G}[N]$ such that $\langle \boldsymbol{\zeta}_1^S, G_k \rangle > \epsilon$. Hence $\mathcal{G}[N]$ is (N, ζ, ϵ) large and by the inductive assumption there exists $N' \in [N]$ such that $\mathcal{G}[N']^{(\omega^\zeta)} \neq \emptyset$. Hence

$$\mathcal{F}[N']^{(\omega^\zeta \cdot n)} = \left[\mathcal{F}[N']^{(\omega^\zeta \cdot (n-1))}\right]^{(\omega^\zeta)} \supset G[N']^{(\omega^\zeta)} \neq \emptyset.$$

This completes the inductive proof of the lemma.

COMPLETION OF THE PROOF OF THE PROPOSITION. Using the previous lemmas, for a given $L \in [M]$, we choose $L \supset L_1 \supset L_2 \supset \cdots \supset L_n \supset \cdots$ such that $s(\mathcal{F}[L_n]) > \omega^\zeta \cdot n$. Then it is easy to see that if N is any set almost contained in L_n for all $n \in \mathbf{N}$, then $s(\mathcal{F}[N]) > \omega^\zeta \cdot n$ for all $n \in \mathbf{N}$, and hence $s(\mathcal{F}[N]) > \omega^{\zeta+1} = \omega^\xi$. The proof of the proposition is complete.

We conclude this section with the following proposition.

2.3.6 PROPOSITION: *Let $\xi < \omega_1$, $M \in [\mathbf{N}]$, $\epsilon > 0$ and $\mathcal{F}$ be an adequate family. Suppose that there exists $L \in [M]$, $L = (m_n)_{n \in \mathbf{N}}$ satisfying the property:*

for every $n \in \mathbf{N}$, $N \in [L]$, $n \leq \min N$

$$\sup_{F \in \mathcal{F}} \left\{ \langle \boldsymbol{\xi}_k^N, F \rangle : k = 1, 2, \ldots, n \right\} > \epsilon.$$

Then there exists $N \in [L]$ such that $s(\mathcal{F}[N]) > \omega^{\xi+1}$.

Proof: Notice that $\mathcal{F}$ satisfies the assumptions of the previous lemma, hence there exists a decreasing sequence $(L_n)_{n \in \mathbf{N}}$ of subsets of L such that $s(\mathcal{F}[L_n]) > \omega^\xi \cdot n$. Now, if N is almost contained in L_n for all $n \in \mathbf{N}$, it is easy to see that $s(\mathcal{F}[N]) > \omega^{\xi+1}$.

THE MAIN RESULTS. We now state and prove the main results.

2.4.1 THEOREM: *For a weakly null sequence $(x_n)_{n \in \mathbf{N}}$ in a Banach space X and $\xi < \omega_1$, exactly one of the following holds:*

(a) *For every $M \in [\mathbf{N}]$ there exists $L \in [M]$ such that for every $P \in [L]$ the sequence $(x_n)_{n\in\mathbf{N}}$ is (P,ξ) summable.*

(b) *There exists $M \in [\mathbf{N}]$, $M = (m_i)_{i\in\mathbf{N}}$, such that $(x_{m_i})_{i\in\mathbf{N}}$ is an $\ell^1_{\xi+1}$ spreading model.*

To prove the theorem we begin with the following lemma.

2.4.2. LEMMA: *Assume that $F = (x_n)_{n\in\mathbf{N}}$ is a weakly null sequence and $\xi < \omega_1$ is such that for every $M \in [\mathbf{N}]$ there exists $N \in [M]$ with $(x_n)_{n\in\mathbf{N}}$ not (N,ξ) summable. Then there exists $\epsilon > 0$ and $L \in [\mathbf{N}]$ such that for every $N \in [L]$*

$$\overline{\lim}\,\|z_n^L\| > \epsilon, \qquad \text{where } z_n^L = \frac{\sum_{k=1}^n \boldsymbol{\xi}_k^L \cdot F}{n}.$$

Proof: We prove it for $M = \mathbf{N}$. The general case is similar. For given $\epsilon > 0$, $n \in \mathbf{N}$, we consider the set

$$A_{\epsilon,n} = \{M \in [\mathbf{N}]\colon \|z_k^M\| \leq \epsilon \forall k \geq n\}.$$

Clearly each $A_{\epsilon,n}$ is a closed set, hence the set $A_\epsilon = \bigcup_{n=1}^\infty A_{\epsilon,n}$ is a Ramsey set. Therefore there exists $L_\epsilon \in [\mathbf{N}]$ such that $[L_\epsilon] \subset A_\epsilon$ or $[L_\epsilon] \subset [\mathbf{N}] \setminus A_\epsilon$.

If there exists some $\epsilon > 0$ and $L \in [\mathbf{N}]$ such that $[L] \subset [\mathbf{N}] \setminus A_\epsilon$ then the lemma has been proved. Assume that this does not occur. Then inductively choose $\mathbf{N} \supset L_1 \supset L_2 \supset \cdots \supset L_n \supset \cdots$ such that $[L_n] \subset A_{1/n}$ and let L be any infinite set almost contained in L_n for all $n \in \mathbf{N}$.

CLAIM: *If $N \in [L]$ then $(x_n)_{n\in\mathbf{N}}$ is (N,ξ) summable.*

Indeed, for any such N and $n \in \mathbf{N}$ there exists $k_n \in \mathbf{N}$ such that for every $k \in \mathbf{N}$ with $k_n \leq k$ we have that $\operatorname{supp}\boldsymbol{\xi}_k^N \subset L_n$. Therefore if $N' = \bigcup_{k\geq k_n} \operatorname{supp}\boldsymbol{\xi}_k^N$ then by the property P.4 we get that $\boldsymbol{\xi}_s^{N'} = \boldsymbol{\xi}_{(s-1)+k_n}^N$. Since $N' \in [L_n]$, there exists s_0 such that for all $s \geq s_0$ we have that $\|z_s^{N'}\| \leq 1/n$. But then there exist large s_1 such that for every $s > s_1$ we have

$$\begin{aligned}\|z_s^N\| &= \left\|\frac{\sum_{i=1}^s \boldsymbol{\xi}_i^N \cdot F}{s}\right\| \\ &= \left\|\frac{\sum_{i=1}^{k_n-1} \boldsymbol{\xi}_i^N \cdot F}{s} + \frac{\sum_{i=1}^s \boldsymbol{\xi}_i^{N'} \cdot F}{s} - \frac{\sum_{i=s+1-k_n}^s \boldsymbol{\xi}_i^{N'} \cdot F}{s}\right\| \\ &< \frac{k_n-1}{s} + \frac{1}{n} + \frac{k_n-1}{s} < \frac{2}{n}.\end{aligned}$$

This proves the Claim and it contradicts our assumptions. Hence there exists $\epsilon > 0$ and $L \in [\mathbf{N}]$ such that $[L]$ is a subset of $[\mathbf{N}] \setminus A_\epsilon$ and this completes the proof of the lemma.

2.4.3 LEMMA: *Let $F = (x_n)_{n\in\mathbf{N}}$ be a weakly null sequence. Suppose that for $\xi < \omega_1$, $M \in [\mathbf{N}]$ and $\epsilon > 0$ we have that for all $N \in [M]$*

$$\overline{\lim}\|z_n^N\| > \epsilon \quad \text{where } z_n^N = \frac{\sum_{i=1}^n \boldsymbol{\xi}_i^N \cdot F}{n}.$$

Then for every $L \in [M]$ we have:

(a) *For every $n \in \mathbf{N}$ there exists $L_n \in [L]$ such that for every $N \in [L_n]$*

$$a_n^N = \sup_{x^* \in B_{X^*}} \min\{x^*(\boldsymbol{\xi}_k^N \cdot F)\colon k = 1, 2, \ldots, n\} > \frac{\epsilon}{2}.$$

(b) *There exists $N \in [L]$, $N = (m_n)_{n\in\mathbf{N}}$ such that, for every $N' \in [N]$ with $m_n \leq \min N'$,*

$$a_n^{N'} > \frac{\epsilon}{2}.$$

Proof: (a) For a given $L \in [M]$ and $n \in \mathbf{N}$ we partition $[L]$ into A_1, A_2 by

$$A_1 = \left\{N \in [L]\colon a_n^N > \frac{\epsilon}{2}\right\}, \quad A_2 = [L] \setminus A_1.$$

The set A_1 is a Ramsey set, hence there exists L_n such that either $[L_n] \subset A_1$ or $[L_n] \subset A_2$. The first case proves part (a) of the Lemma. We show that the second case does not occur.

Indeed, if $[L_n] \subset A_2$ then we get that for $k_1 < k_2 < \cdots < k_n$ there exists $N \in [L_n]$ such that $\boldsymbol{\xi}_{k_1}^{L_n} = \boldsymbol{\xi}_1^N, \ldots, \boldsymbol{\xi}_{k_n}^{L_n} = \boldsymbol{\xi}_n^N$ and, since $N \in A_2$, $a_n^N \leq \epsilon/2$. Choose s large such that there exists $x^* \in B_{X^*}$ with

$$x^*\left(\frac{\sum_{i=1}^s \boldsymbol{\xi}_i^{L_n} \cdot F}{s}\right) > \epsilon.$$

Then from the choice of s we get that

$$\#\left\{i\colon i \leq s, x^*(\boldsymbol{\xi}_i^{L_k} \cdot F) > \epsilon/2\right\} \geq n.$$

But then there exists $N \in [L_n]$ with $a_N^n > \epsilon/2$, a contradiction, and the proof of part (a) is complete.

(b) Choose, inductively, a decreasing sequence $(L_n)_{n\in\mathbf{N}}$ such that $L_n \in [L]$ and L_n satisfies the requirement for the number n of part (a). It is clear that any N almost contained in L_n for all $n \in \mathbf{N}$ is the desired set.

Next we will prove two lemmas that will help us to reduce the proof of the theorem to the case of the sequence $(\pi_n)_{n\in L}$ of the natural coordinate projections of $\{0,1\}^{\mathbf{N}}$ acting on an adequate family $\boldsymbol{\mathcal{F}}$ of finite subsets of $\mathbf{N}$.

2.4.4 Definition: Let D be a weakly compact subset of $c_0(\mathbf{N})$ and $\delta > 0$. We set

$$\mathcal{F}_\delta = \{F \subset \mathbf{N} : \exists f \in D \text{ with } f(n) \geq \delta \forall n \in F\}.$$

2.4.5 Remark: The weak compactness of D implies that $\mathcal{F}_\delta$ is an adequate family of finite subsets of $\mathbf{N}$.

2.4.6 Notation: As we denoted in 2.1.5, $\mathcal{F}_\delta[N] = \{F \in \mathcal{F}_\delta : F \subset N\}$.

The next Lemma is a consequence of Lemma 1.2.

2.4.7 LEMMA: *Let D be a weakly compact subset of $c_0(\mathbf{N})$. Then for every $\delta > 0$, $\epsilon > 0$ and $M \in [\mathbf{N}]$ there exists $N \in [M]$ such that for every $F \in \mathcal{F}_\delta[N]$ there exists $f \in D$ such that*

(i) $\min\{f(n) : n \in F\} \geq (1 - \epsilon)\delta$,

(ii) $\sum_{n \in N \setminus F} |f(n)| \leq \epsilon \cdot \delta$.

Proof: From Lemma 1.2 for every $M \in [\mathbf{N}]$ there exists $N \in [M]$ such that for every $k \in \mathbf{N}$, $F \in \mathcal{F}_\delta[N]$ with $\max F \leq k$ there exists $f_k \in D$,

(i) $\min\{f_k(n) : n \in F\} > (1 - \epsilon)\delta$,

(ii) $\sum_{\substack{n=1 \\ n \notin F}}^{k} |f_k(n)| < \epsilon \cdot \delta$.

The desired f is the weak limit of any weakly convergent subsequence of $(f_k)_{k \in \mathbf{N}}$.

2.4.8 LEMMA (Reduction Lemma): *Let $H = (x_n)_{n \in \mathbf{N}}$ be a weakly null sequence in a Banach space with $\|x_n\| \leq 1$. Then for every $\delta > 0$ and $\epsilon > 0$ there exists an adequate family $\mathcal{F}$ of finite subsets of $\mathbf{N}$ and a function $f : B_{X^*} \longrightarrow \mathcal{F}$ such that:*

For every $M \in [\mathbf{N}]$ there exists $N \in [\mathbf{N}]$ satisfying the following properties:

(a) *If $A \in S^+_{\ell^1}$, $\operatorname{supp} A \subset N$, then for every $x^* \in B_{X^*}$ with $\langle x^*, A \cdot H \rangle > \delta$ we have $\langle A, f(x^*) \rangle > \delta^2/4$.*

(b) *If $A \in S^+_{\ell^1}$ with $\operatorname{supp} A \subset N$ and $F \in \mathcal{F}$ such that $\langle A, F \rangle \geq \epsilon$, then $\|A \cdot H\| \geq \epsilon \cdot \delta/4$.*

Proof: We start by noticing that if $A \in S^+_{\ell^1}$ and $x^* \in B_{X^*}$ are such that $A = (a_n)_{n \in \mathbf{N}}$ and $x^*(A \cdot H) > \delta$, then for $F = \{n \in \mathbf{N} : x^*(x_n) > \delta/2\}$ we get that $\sum_{n \in F} a_n > \delta/2$. Hence $\langle A, F \rangle > \delta^2/4$.

Since $(x_n)_{n \in \mathbf{N}}$ is a weakly null sequence, the set

$$D = \{(x^*(x_n))_{n \in \mathbf{N}} : x^* \in B_{X^*}\}$$

is a weakly compact subset of $c_0(\mathbf{N})$. Applying Lemma 2.4.7 for D, $\delta/2$, $\epsilon/4$ and $M \in [\mathbf{N}]$ we find $N \in [M]$ satisfying properties (i) and (ii) of that lemma. We let $\mathcal{F}$ be the adequate family defined as $\mathcal{F}_{\delta/2}$. We also define $f : B_{X^*} \longrightarrow \mathcal{F}$ by the rule $f(x^*) = \{n \in \mathbf{N}\colon x^*(x_n) > \delta/2\}$. Using our note at the beginning of the proof, we get that property (a) holds for every $N \in [M]$. To see property (b), suppose that $A \in S_{\ell^1}$ with $\operatorname{supp} A \subset N$ and $F \in \mathcal{F}$ such that $\langle A, F\rangle \geq \epsilon$. Then we may assume that $F \subset N \cap \{n \in \mathbf{N}\colon a_n > 0\}$ and, by the definition of $\mathcal{F}$, there exists $G \in \mathcal{F}_{\delta/2}$ such that $F \subset G \cap N$. Then there exists $x^* \in B_{X^*}$ such that

(i) $\min\{x^*(x_n)\colon n \in F\} \geq \left(1 - \frac{\epsilon}{4}\right)\frac{\delta}{2}$,

(ii) $\sum_{n \notin F} |x^*(x_n)| \leq \frac{\epsilon}{4}\frac{\delta}{2}$.

From (i), (ii) and the fact that $\langle A, F\rangle \geq \epsilon$ we get that $\|A \cdot H\| > \epsilon \cdot \delta/4$. The proof is complete.

Proof of the theorem: We prove first that the negation of (a) implies (b). Suppose that $\|x_n\| \leq 1$ and for a given $\xi < \omega_1$ the case (a) does not occur. Then from Lemma 2.4.2 there exist $M \in [\mathbf{N}]$ and $\delta > 0$ such that $\overline{\lim}\|z_n^L\| > 2\delta$ for all $L \in [M]$. Going to a subset of M if necessary, we may assume that part (b) of Lemma 2.4.3 is also satisfied for M with $\epsilon/2$ replaced by δ.

Consider the family $\mathcal{F}$ defined in Lemma 2.4.8 for the sequence $(x_n)_{n\in\mathbf{N}}$ and the number δ. Let $N \in [M]$ such that (a) and (b) in Lemma 2.4.8 are satisfied. Property (a) in connection with the fact that N satisfies the conclusion of Lemma 2.4.3 shows that the assumptions of Proposition 2.3.6 are fulfilled. Hence there exists $N' \in [N]$ such that $s(\mathcal{F}[N']) > \omega^{\xi+1}$. From Theorem 2.2.6 there exists $N_1 \in [N']$ such that $N_1 = (m_i)_{i\in\mathbf{N}}$ and for every $F \in \mathcal{F}_{\xi+1}$ the set $\{m_i\colon i \in F\} \in \mathcal{F}$.

CLAIM: *For every $F \in \mathcal{F}_{\xi+1}$, $\|\sum_{i\in F} a_i x_{m_i}\| \geq \frac{\delta}{8}\sum_{i\in F}|a_i|$.*

Indeed, by standard arguments, it is enough to show it for $(a_i)_{i\in F} \in S_{\ell^1}$. If $(a_i)_{i\in F} \in S_{\ell^1}$ then either $\sum\{a_i\colon i \in F, a_i > 0\} \geq \frac{1}{2}$ or $\sum\{a_i\colon i \in F, a_i < 0\} \leq -\frac{1}{2}$. We assume that the first case occurs. Otherwise we consider $(b_i)_{i\in F}$ such that $b_i = -a_i$ for all $i \in F$. Set $F' = \{i \in F\colon a_i > 0\}$; then clearly $\langle A, F'\rangle \geq \frac{1}{2}$ and hence $\|A \cdot (x_n)_{n\in\mathbf{N}}\| > \delta/8$, which proves the claim. The proof is complete.

We now show that parts (a) and (b) of Theorem 2.4.1 are mutually exclusive.

2.4.9 PROPOSITION: *Let $(x_n)_{n\in\mathbf{N}}$ be a weakly null sequence in a Banach space X. If $\xi < \omega_1$, $M \in [\mathbf{N}]$ and $\delta > 0$ are such that $M = (m_i)_{i\in\mathbf{N}}$ and*

$$\|\sum_{i\in F} a_i x_{m_i}\| \geq \delta \cdot \sum_{i\in F}|a_i| \quad \text{for every } F \in \mathcal{F}_{\xi+1},$$

then there exists $L \in [M]$ such that for every $P \in [L]$, $(x_n)_{n\in\mathbf{N}}$ is not (P,ξ) summable.

Proof: Consider the adequate family $\boldsymbol{\mathcal{F}}$ defined in the Reduction Lemma (Lemma 2.4.8) for the sequence $(x_n)_{n\in\mathbf{N}}$, the number δ in our assumptions and $\epsilon = \frac{1}{2}$ (Proposition 2.1.10). Find $N \in [M]$ such that conditions (a), (b) of the Reduction Lemma are fulfilled. Denote by $\left(\left[\boldsymbol{\xi}^M\right]_n^P\right)_{n\in\mathbf{N}}$, $P \in [N]$, the summability methods defined by the rule $\left[\boldsymbol{\xi}^M\right]_n^P = (a_{m_i})_{i\in\mathbf{N}}$ where $\boldsymbol{\xi}_n^P = (a_i)_{i\in\mathbf{N}}$ and $a_{m_i} = a_i$. Then $\boldsymbol{\mathcal{F}}$ is $(N, (\boldsymbol{\xi}+1)^M, \delta^2/4)$ large, hence there exists $L' \in [\mathbf{N}]$ such that $\boldsymbol{\mathcal{F}}_{\xi+1}^{L'}$ is a subfamily of $\boldsymbol{\mathcal{F}}$, and hence by Proposition 2.1.11 there exists $L'' \in [L']$ such that for every $P \in [L'']$, $n \in \mathbf{N}$ there exists $G \in \boldsymbol{\mathcal{F}}_{\xi+1}^{L'}$ such that $\langle (\boldsymbol{\xi}+\mathbf{1})_n^P, G\rangle > \frac{1}{2}$. Choose, as in Lemma 2.4.3 (part (b)), an $L \in [L'']$ such that $L = (l_n)_{n\in\mathbf{N}}$ and, for every $P \in [L]$ and $n \leq k_1 < k_2 < \cdots < k_n$, there exists $G \in \boldsymbol{\mathcal{F}}$ such that $\left\langle \boldsymbol{\xi}_{k_i}^P, G\right\rangle > \frac{1}{4}$. Then by part (b) of the Reduction Lemma there exists $x^* \in B_{X^*}$ such that $\left\langle \boldsymbol{\xi}_{k_i}^P \cdot H, X^*\right\rangle > \delta/16$ where $H = (x_n)_{n\in\mathbf{N}}$ and $(k_i)_{i=1}^n, P$ are as above. It is clear now that for every $P \in [L]$ the sequence $(x_n)_{n\in\mathbf{N}}$ is not (P, ξ) summable.

2.4.10 Remark: The above Proposition immediately shows that parts (a) and (b) in Theorem 2.4.1 are mutually exclusive.

For the sequel we need the following result proved in [Al–Ar].

2.4.11 Proposition: *Let X be a Banach space and $(x_n)_{n\in\mathbf{N}}$ a weakly null sequence in X.*

(a) *There exists $\xi < \omega_1$ such that for all $\zeta < \omega_1$, $\xi \leq \zeta$, $(x_n)_{n\in\mathbf{N}}$ does not contain a subsequence which is an ℓ_ζ^1 spreading model.*

(b) *If ℓ^1 does not embed into X then there exists $\xi < \omega_1$ such that for every $\zeta < \omega_1$, $\xi \leq \zeta$ and any bounded sequence $(x_n)_{n\in\mathbf{N}}$ there is no subsequence of $(x_n)_{n\in\mathbf{N}}$ which is an ℓ_ζ^1 spreading model.*

Sketch of proof: The proof of (a) follows from the fact that

$$\mathcal{T}_\epsilon = \left\{(x_{n_1}, \ldots, x_{n_k}) : \left\|\sum_{i=1}^k a_i x_{n_i}\right\| \geq \epsilon \cdot \sum_{i=1}^k |a_i|\right\}$$

ordered in the usual manner is a well-founded tree. If not, the sequence $(x_n)_{n\in\mathbf{N}}$ should contain a subsequence equivalent to the unit vector basis of ℓ^1 and that contradicts the weak nullness of $(x_n)_{n\in\mathbf{N}}$. Therefore the height of $\mathcal{T}_\epsilon$, denoted by $o(\mathcal{T}_\epsilon)$, is a countable ordinal ξ_ϵ. Further, if $(x_n)_{n\in\mathbf{N}}$ has a subsequence that is an ℓ_ξ^1 spreading model with constant $\delta_\xi > \epsilon$ then $\omega^\xi \leq \xi_\epsilon$. So if $\xi_0 = \sup\{\xi_\epsilon, \epsilon > 0\}$ then every $\xi < \omega_1$ such that $(x_n)_{n\in\mathbf{N}}$ has a subsequence which in an ℓ_ξ^1 spreading model should satisfy $\omega^\xi \leq \xi_0$ and this proves the result for part (a).

The proof of part (b) is the same and uses the technique developed by Bourgain [B].

THE BANACH – SAKS INDEX.

2.4.12 Definition: Let X be a Banach space and $(x_n)_{n\in\mathbf{N}}$ a weakly null sequence in X.

(a) The Banach–Saks index of $(x_n)_{n\in\mathbf{N}}$ denoted by $\mathrm{BS}[(x_n)_{n\in\mathbf{N}}]$ is the least ordinal ξ such that there is no subsequence of $(x_n)_{n\in\mathbf{N}}$ which is an ℓ^1_ξ spreading model.

(b) If X is a Banach space not containing $\ell^1(\mathbf{N})$, then we denote by $\mathrm{BS}[X]$ the least ordinal ξ such that no bounded sequence $(x_n)_{n\in\mathbf{N}}$ in X is an ℓ^1_ξ spreading model.

2.4.13 THEOREM: *Let $H=(x_n)_{n\in\mathbf{N}}$ be a weakly null sequence with $\mathrm{BS}[(x_n)_{n\in\mathbf{N}}] = \xi$. Then ξ is the unique ordinal satisfying the following:*

(a) *For every $M \in [\mathbf{N}]$ there exists $L \in [M]$ such that for every $P \in [L]$, $\lim_{n\in\mathbf{N}} \|\boldsymbol{\xi}^P_n \cdot H\| = 0$.*
(b) *For every $\zeta < \xi$ there exists $L_\zeta \in [\mathbf{N}]$ such that $L_\zeta = (n_i)_{i\in\mathbf{N}}$ and $(x_{n_i})_{i\in\mathbf{N}}$ is an ℓ^1_ζ spreading model.*
(c) *If $\xi = \zeta + 1$ there exists $\epsilon > 0$ and $L \in [\mathbf{N}]$ such that for all $P \in [L]$, $\left\|\boldsymbol{\zeta}^P_n \cdot H\right\| > \epsilon$ and $\left(\boldsymbol{\zeta}^P_n \cdot H\right)_{n\in\mathbf{N}}$ is Cesaro summable.*

Proof: (a) For $L \in [\mathbf{N}]$ and $n \in \mathbf{N}$ we define a partition of $[L]$ into sets A, B by the rule $A = \{P\colon \|\boldsymbol{\xi}^P_1 \cdot H\| \le \epsilon\}$ and $B = [L] \setminus A$. It is easy to see that A is a closed subset of $[L]$, hence by Theorem 0.1 there exists $N \in [L]$ such that either $[N] \subset A$ or $[N] \subset B$. If the second case holds then by the Reduction lemma we get that $(x_n)_{n\in\mathbf{N}}$ has a subsequence which is an ℓ^1_ξ spreading model, a contradiction. Hence $[N] \subset A$. Choose, inductively, $M \supset L_1 \supset L_2 \supset \cdots \supset L_n \supset \cdots$ such that for every $P \in [L_n]$, $\|\boldsymbol{\xi}^P_1 \cdot H\| < 1/n$ and set $L = (l_n)_{n\in\mathbf{N}}$ such that $l_n \in L_n$. Then it is easy to see L satisfies the conclusion of the first part of the theorem.

(b) This follows from the definition of ξ.

(c) Suppose now that $\xi = \zeta + 1$. Then, by the definition of $\mathrm{BS}[(x_n)_{n\in\mathbf{N}}]$, there exists $M \in [\mathbf{N}]$ such that $(x_n)_{n\in M}$ is an ℓ^1_ζ spreading model. Then by part (a) of Theorem 2.4.1 there exists $N \in [M]$ such that for every $P \in [N]$ $(x_n)_{n\in\mathbf{N}}$ is (P,ξ) summable and, finally, from Proposition 2.4.9 there exist $L \in [N]$ and $\epsilon > 0$ such that for every $P \in [L]$ and $n \in \mathbf{N}$, $\|\boldsymbol{\zeta}^P_n \cdot H\| \ge \epsilon$. This proves part (c) and the proof is complete.

2.4.14 Remark: (i) The first part of the above Theorem is satisfied by any normalized weakly null sequence in Tsirelson's space. Any such sequence has Banach–Saks index equal to ω.

(ii) The third part gives a complete answer in the following question posed by the first-named author: For what weakly null sequences does there exist a sequence $(y_n)_{n\in\mathbf{N}}$ of block convex combinations such that $\|y_n\| > \epsilon$ and $(y_n)_{n\in\mathbf{N}}$ is Cesaro summable.

We conclude this Section with the following corollaries. Their proofs follow easily from the previous theorems.

2.4.15 COROLLARY: *For every separable reflexive Banach space X there exists a unique ordinal $\xi < \omega_1$ such that:*

(i) *For all ordinals $\zeta \geq \xi$ the space X has ζ-BS.*

(ii) *For every $\zeta < \xi$ the space X fails ζ-BS.*

2.4.16 COROLLARY: *If X is a separable Banach space not containing ℓ^1 isomorphically then there exist a unique ordinal $\xi < \omega_1$ such that:*

(i) *For all ordinals $\zeta \geq \xi$ the space X has w ζ-BS.*

(ii) *For every $\zeta < \xi$ the space X fails w ζ-BS.*

THE ANTI-UNIFORM CONVERGENCE INDEX. Let K be a compact space and $f_n: K \longrightarrow \mathbf{R}$, $n \in \mathbf{N}$ a sequence of continuous functions such that $f_n\rangle 0$ pointwise on K. If P is any closed non-empty subset of K and $\epsilon > 0$ we set

$$P'((f_n),\epsilon) = \{x \in P \colon \forall \text{ nbhd's } V \text{ of } x \text{ the set } \{n \in \mathbf{N}\colon \|f_n|_{P\cap V}\| \leq \epsilon\} \text{ is finite}\}.$$

It is clear that the set $P'((f_n),\epsilon)$ is obtained from P by deleting the relatively open sets in P on which some subsequence of (f_n) is ϵ-uniformly convergent. We define by induction the iterates $P^{\xi}((f_n),\epsilon)$ for each ordinal ξ, and let

$$\mathrm{auc}((f_n),\epsilon,P) = \begin{cases} \text{least } \xi \text{ with } P^{\xi}((f_n),\epsilon) = \emptyset, \text{ if such a } \xi \text{ exists,} \\ \omega_1, \text{ otherwise.} \end{cases}$$

We also set

$$\mathrm{auc}((f_n),\epsilon) = \mathrm{auc}((f_n),\epsilon,K) \quad \text{and} \quad \mathrm{auc}((f_n)) = \sup_{\epsilon>0} \mathrm{auc}((f_n),\epsilon)$$

(cf. the definition of "convergence rank" in [K–L], p. 212).

2.5.1 Remarks: (1) It is clear that for a closed non-empty subset P of K, the set $P'((f_n),\epsilon)$ is a closed subset of P.

(2) If (f'_n) is any subsequence of (f_n) then it is easy to see that

$$\operatorname{auc}((f'_n)) \geq \operatorname{auc}((f_n)).$$

(3) If K is compact metric, $\epsilon > 0$, $M \in [\mathbf{N}]$ and $t \in K'((f_n),\epsilon)$, then a simple diagonal argument shows that there exist $(t_n)_{n\in\mathbf{N}} \subset K$ and $L = (l_n) \in [M]$ such that $t_n \to t$, $t_n \neq t_m$ for $m < n$ and $|f_{l_n}(t_n)| > \epsilon$ for every $n = 1, 2, \ldots$.

2.5.2 PROPOSITION: *Let (f_n) be a sequence of continuous functions on the compact metric space K such that $f_n \rangle 0$ pointwise on K. Then there is $\xi < \omega_1$ such that* $\operatorname{auc}((f'_n)) < \xi$ *for every subsequence (f'_n) of (f_n).*

Proof: For a closed nonempty $P \subset K$ and $\epsilon > 0$ we set

$$P'_\epsilon := \Big\{x \in P\colon \forall \text{ nhhd's } V \text{ of } x \text{ the set } \{n \in \mathbf{N}\colon \|f_n|_{P\cap V}\| > \epsilon\} \text{ is infinite}\Big\}.$$

It then follows from Baire's Category Theorem that P'_ϵ is a closed nowhere dense subset of P. If (f'_n) is any subsequence of (f_n) then it is clear that

$$P'((f'_n),\epsilon) \subset P'_\epsilon;$$

furthermore, by induction one can prove that $P^\xi((f'_n),\epsilon) \subset P^\xi_\epsilon$ for every ordinal ξ. Since K is a compact metric space and the family $(K^\xi_\epsilon)_{\xi<\omega_1}$ is decreasing and consists of closed subsets of K, there is $\xi(\epsilon) < \omega_1$ such that $K^{\xi(\epsilon)}_\epsilon = \emptyset$. It is clear that $\xi = \sup\{\xi_{1/n}\colon n \in \mathbf{N}\}$ is the desired countable ordinal.

Let K be a compact metric space and $f_n\colon K \longrightarrow \mathbf{R}$, $n \in \mathbf{N}$ a sequence of continuous functions on K such that $f_n \rangle 0$ pointwise on K. For every $\epsilon > 0$ we set $\mathcal{F}_\epsilon = \{A \subset \mathbf{N}\colon \exists t \in K \text{ with } |f_n(t)| \geq \epsilon\ \forall n \in A\}$. It is easy to see that $\mathcal{F}_\epsilon$ is an adequate family of (finite) subsets of $\mathbf{N}$.

2.5.3 THEOREM: *Assume that* $\operatorname{auc}((f_n),\epsilon) > \xi$ *for some $\xi < \omega_1$ and some $\epsilon > 0$. Then we have $s(\mathcal{F}_\epsilon[L]) > \xi$ for every $L \in [\mathbf{N}]$.*

For the proof of this theorem we need the following.

2.5.4 LEMMA: *Let $1 \leq \xi < \omega_1$, $\epsilon > 0$, $t \in K^\xi((f_n),\epsilon)$ and $A \in [\mathbf{N}]^{<\omega}$ such that $|f_n(t)| > \epsilon$ for each $n \in A$. Then $A \in \mathcal{F}^{(\xi)}_\epsilon$.*

Proof: We proceed by induction on ξ.

CASE 1: $\xi = \zeta + 1$. Let $t_0 \in K^{\zeta+1}((f_n), \epsilon)$ and $A \in [\mathbf{N}]^{<\omega}$ be such that $|f_m(t_0)| > \epsilon$ for every $m \in A$. Since for each $m \in A$ the function f_m is continuous and $|f_m(t_0)| > \epsilon$, it follows that there is a nbhd V of t_0 such that

$$(1) \qquad |f_m(t)| > \epsilon \quad \text{for } t \in V \text{ and } m \in A.$$

Let $N \in [\mathbf{N}]$. Since $t_0 \in K^{\zeta+1}((f_n), \epsilon)$ there exist $M \in [N]$, $M = (k_n)_{n\in\mathbf{N}}$ and a sequence $(t_n)_{n\in\mathbf{N}} \subset V \cap K^{\zeta}((f_n), \epsilon), t_n \to t_0$, $t_n \neq t_m$ for $m < n$ such that

$$|f_{k_n}(t_n)| \geq \epsilon \quad \text{for } n = 1, 2, \ldots.$$

It follows immediately that for $n = 1, 2, \ldots$ we have

$$|f_m(t_n)| \geq \epsilon \quad \text{for every } m \in A \cup \{k_n\}.$$

Therefore from the inductive assumption we get that

$$A \cup \{k_n\} \in \mathcal{F}_\epsilon^\zeta \quad \text{for every } n = 1, 2, \ldots.$$

We conclude that for every $N \in [\mathbf{N}]$, A is a cluster point of $\mathcal{F}_\epsilon^\zeta[A \cup N]$, hence $A \in \mathcal{F}_\epsilon^{\zeta+1}$.

CASE 2: *ξ is a limit ordinal.* This is obvious.

The proof of the Lemma is complete.

Proof of the Theorem: We proceed by induction on ξ. We notice that the assertion is clear for $\xi = 0$.

CASE 1: $\xi = \zeta + 1$. Since $\mathrm{auc}((f_n), \epsilon) > \xi$ we get that $K^{\zeta+1}((f_n)_{n\in L}, \epsilon) \neq \emptyset$ for every $L \in [\mathbf{N}]$. So let $L \in [\mathbf{N}]$ and $t_0 \in K^{\zeta+1}((f_n)_{n\in L}, \epsilon)$. It then follows from the definition of auc-index that there is $(t_n) \supset K^{\zeta}((f_n))_{n\in L}, \epsilon)$ and $M \in [L]$, $M = (k_n)_{n\in\mathbf{N}}$ such that, $t_n \to t_0$ and $|f_{k_n}(t_n)| > \epsilon$ for $n = 1, 2, \ldots$. So we get from the previous lemma that $\{m\} \in \mathcal{F}_\epsilon^\zeta[L]$ for all $m \in M$, hence $\mathcal{F}_\epsilon^{\zeta+1}[L] \neq \emptyset$ (because $\emptyset \in \mathcal{F}_\epsilon^{\zeta+1}[L]$) and thus

$$s(\mathcal{F}_\epsilon[L]) > \zeta + 1 = \xi.$$

CASE 2: *ξ is a limit ordinal.* Let $\zeta < \xi$; then we have that $\mathrm{auc}((f_n), \epsilon) > \xi > \zeta$, hence by the inductive assumption $s(\mathcal{F}_\epsilon[L]) > \zeta$ for every $L \in [\mathbf{N}]$. Since ξ is a limit ordinal we get immediately that $s(\mathcal{F}_\epsilon[L]) > \xi$ for every $L \in [\mathbf{N}]$.

2.5.5 PROPOSITION: *Let $H = (x_n)_{n\in\mathbf{N}}$ be a weakly null sequence and $\delta > 0$. Set*

$$D = \{(x^*(x_n))_{n\in\mathbf{N}} : \|x^*\| \leq 1\}$$

and let $\mathcal{F}_\delta$ be the family corresponding to the set D and the number δ. If there exists $M \in [\mathbf{N}]$ such that $\mathcal{F}_\xi^M$ is a subfamily of $\mathcal{F}_\delta$, then $(x_n)_{n\in\mathbf{N}}$ has a subsequence which is an ℓ_ξ^1 spreading model.

Proof: From Lemma 2.4.8 (b) for $\epsilon = 1/2$, there exists $N \in [M]$ such that for every $A = (a_n)_{n\in\mathbf{N}} \in S_{\ell^1}^+$ with $\operatorname{supp} A \in \mathcal{F}_\xi^M[N]$, $\|A \cdot H\| > \delta/4$. It follows that $(x_n)_{n\in\mathbf{N}}$ is an ℓ_ξ^1 spreading model.

2.5.6 THEOREM: *Suppose that $(x_n)_{n\in\mathbf{N}}$ is a weakly null sequence and $\operatorname{auc}((f_n)_{n\in\mathbf{N}}) > \omega^\xi$. Then there exists $M \in [N]$ such that $(x_n)_{n\in M}$ is an ℓ_ξ^1 spreading model.*

Proof: This follows from Theorem 2.2.6 and the above Proposition.

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